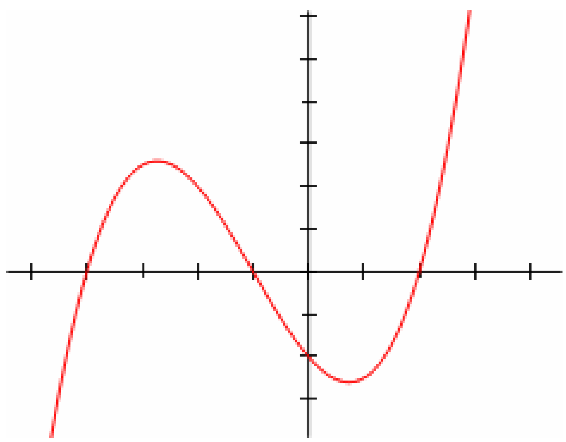
Complement: Cardano's Formulae



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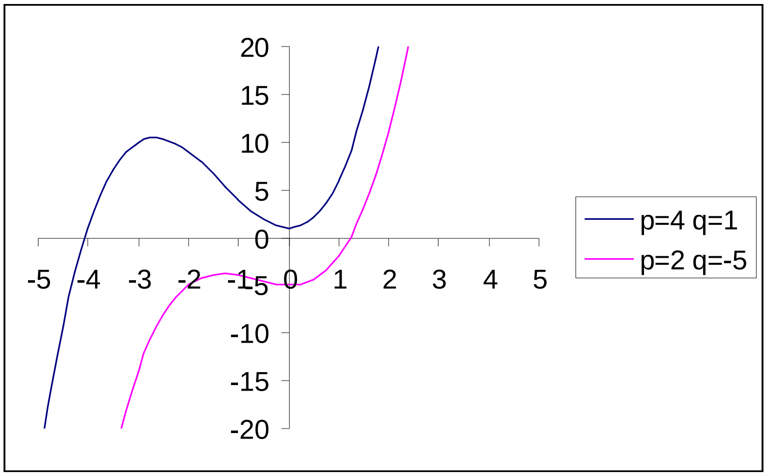
Complement : Cardano's Formulae

Cardano's formulae give the roots of cubic (third degree polynomial) P(X).
We only consider in this document the cubic functions $P(X) = X^3 + pX + q$ since all cubic functions can be re-arranged into looking like P(X).



We seek roots of P(X) by writing $X = h + k$
h and k must therefore satisfy

$$X^3 + pX + q = h^3 + k^3 + (3hk + p)(h + k) + q$$

Cardano's Trick

If we can find h and k such as $\begin{cases} h^3 + k^3 = -q \\ hk = -p/3 \end{cases}$ then we have found a root of the cubic.

The conditions are rewritten as $\begin{cases} h^3 + k^3 = -q \\ h^3k^3 = -p^3/27 \end{cases}$ which suggests the change $\begin{cases} u = h^3 \\ v = k^3 \end{cases}$

We eventually get the system in (u,v) $\begin{cases} u + v = -q \\ uv = -p^3/27 \end{cases}$

u and v are roots of the second degree polynomial $Y^2 + qY - p^3/27$

The discriminant of the polynomial $\Delta = q^2 + 4p^3/27$

Case 1: $\Delta > 0$

$Y^2 + qY - p^3/27$ has two real roots $u = \dfrac{-q + \sqrt{\Delta}}{2}$ and $v = \dfrac{-q - \sqrt{\Delta}}{2}$

We have three possible values for h $h_0 = \left(\dfrac{-q + \sqrt{\Delta}}{2}\right)^{1/3}$, jh_0 and $j^2 h_0$ (where j is the cubic root of unity $j = e^{i2\pi/3}$) for which the corresponding values of k are $k_0 = \left(\dfrac{-q - \sqrt{\Delta}}{2}\right)^{1/3}$, $j^2 k_0$ and jk_0.

The equation is solved as we have three possible (one real, two complex) combinations $X = h + k$:

$$X_1 = h_0 + k_0,\ X_2 = jh_0 + j^2 k_0 \text{ and } X_3 = j^2 h_0 + jk_0 = \overline{X_2}$$

Case 2 (Limit Case): $\Delta = 0$

$Y^2 + qY - p^3/27$ has a double real root $u = v = \dfrac{-q}{2}$

We have three possible values for h $h_0 = \left(\dfrac{-q + \sqrt{\Delta}}{2}\right)^{1/3}$, jh_0 and $j^2 h_0$ for which the corresponding values of k are $k_0 = h_0$, $j^2 h_0$ and jh_0.

The equation is solved as we have two real roots $X = h + k$ of the cubic:

$$X_1 = 2h_0 \text{ and } X_2 = jh_0 + j^2 h_0 = -h_0$$

Case 3 $\Delta < 0$,

$Y^2 + qY - p^3/27$ has two roots $u = \frac{-q + i\sqrt{-\Delta}}{2}$ and $v = \frac{-q - i\sqrt{-\Delta}}{2}$.

We have three possible values for h h_0 (one of the cubic root of u), jh_0 and j^2h_0 . The corresponding values of k are $k_0 = \overline{h_0}$, $j^2\overline{h_0}$ and $j\overline{h_0}$.

The equation is solved as we have three <u>real</u> possible combinations $X = h + k$:

$$h_0 + \overline{h_0} = 2\,\mathrm{Re}(h_0) \ ,$$

$$jh_0 + j^2\overline{h_0} = 2\,\mathrm{Re}(jh_0)$$

and

$$j^2h_0 + j\overline{h_0} = 2\,\mathrm{Re}(j^2h_0)$$