Model Analysis

Introduction

- To investigate the behaviours of a given 2-D ODE systems
- Use Lotka-Volterra Equation to illustrate the analysis method.

Lotka-Volterra Equation

- Earliest known model proposed for a predator-prey system
- Natural oscillation of population of predator and prey
- Modelled by a pair of first order, non-linear, differential equations, describing the dynamics of biological systems in which the predator and prey interact

Objective:

- To study and understand the dynamics of Lotka-Volterra system
- To define an approach to study a given set of parameters

Assumption made by Lotka-Volterra system

- The prey has unlimited exponential growth
- The prey does not die itself, its death is only caused by predator
- The predator's growth is upon its killing of prey, and it will have its natural decay

ODE (Ordinary Differential Equation)

$$\frac{dx}{dt} = ax - bxy$$

$$\frac{dy}{dt} = cxy - dy$$

x	The population of prey at time t
y	The population of predator at time t
a	Rate constant on how fast prey reproduce themselves
b	Rate constant on how fast prey killed by predator
c	Rate constant on how fast predator reproduce when fed from prey
d	Rate constant on how fast predator dies

- The production rate of prey is directly proportional to the prey population, i.e. prey will have exponential growth itself. The more number of preys, the more production of preys.
- The death rate of prey is proportional to the product of prey and predator population, i.e. it modeled the death of prey during the encounter of predator.
- The production rate of the predator is proportional to the product of prey and predator population, i.e. it modeled the growth of predator when it is fed upon prey.
- The death rate of predator is directly proportional to the predator population, i.e. predator will have exponential decay itself. The more number of predators, the more death of predators occurred.

Stationary Points

- Stationary Points in the dynamic system are points when the overall system is ceased to change anymore over the time, i.e. when the system starts at/reaches the stationary point, it will stay forever in the absence of external force.
- By definition of the stationary points, they could be found mathematically by solving simultaneous equations dx/dt=0 & dy/dt=0
- In this Lotka-Volterra model:
  dx/dt=ax-bxy=0 → x=0 or y=a/b
  dy/dt=cxy-dy=0 → y=0 or x=d/c
- Thus there are two stationary points. (0, 0) & (d/c, a/b)
- If a stationary point is stable, any small perturbation from the stationary point will be brought back to it, otherwise it is unstable. I.e. the stationary point is resilient to noise.
- Jacobian matrix is used to determine whether these points are stable.

Jacobian Analysis

- Jacobian matrix is a linear approximation to a differentiable function near a given point.
- Jacobian matrix is formed by the partial derivatives of the ODE functions. In this way, we could approximately decouple the interfering parameters; such linearization allows us to study each parameter independently.

$$J_F(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

- In our Lotka-Volterra model, the Jacobian matrix is $\begin{bmatrix} a-by & -bx \\ cy & cx-d \end{bmatrix}$
- The benefit of using Jacobian matrix is that we can use the eigenvalues of the matrix to decouple the parameters, such as following:

$$\begin{bmatrix} \lambda_1 & 0 & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . \\ . & 0 & . & 0 & . \\ . & . & 0 & . & 0 \\ 0 & . & . & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ . \\ . \\ z_n \end{bmatrix}$$

- In x-y two dimensional systems, the eigenvalues are correspondent to the local change of the point, i.e. it can be approximated that $\partial x = e^{\lambda_1 t} x$ and $\partial y = e^{\lambda_2 t} y$
- Hence, in the dynamic systems, the behaviour of the system near a stationary point can be determined by the nature of the eigenvalues.

Stability analysis (eigenvalue)

- Taking $\partial x = e^{\lambda_1 t} x$ as a reference, we can see that the eigenvalues is within the form of $e^{\lambda t}$, thus

- The real parts of eigenvalues are responsible for the magnitude of changes
  - If it is positive, the magnitude of changes will have an exponential growth, i.e. the point is going to move further away in x direction – **unstable** characteristic
  - If it is negative, the change will an exponential decay, i.e. the point is going moving closer and closer to the stationary point in x direction – **stable** characteristic
- The imaginary parts of eigenvalues are responsible for the spins
  - By Euler's formula, $e^{i\theta t} = \cos\theta t - i\sin\theta t$, the mean of cosine or sine is zero, hence the imaginary parts will not have any contribution to the magnitude of changes
  - However, it will affect how fast the point is going to spin around.

- Thus the stability of a stationary point can be characterized by the real parts of eigenvalues
  - If **ALL** the real parts of eigenvalues are **negative**, then the point is **stable**
  - Otherwise it will be unstable.
  - Special case when the real parts are **zero**, then the point is a **center**, all the points around will oscillate with respect to the stationary point.
- Hence, in our 2-D system the stationary points can be characterized as follow:
  - If both the real parts of eigenvalues are negative, it is stable.
  - Otherwise it is unstable.
  - Special case of unstable: If both the real parts of eigenvalues are zero, it is a center.
- In Lotka-Volterra system, for the stationary point (0, 0) the Jacobian matrix is $\begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$, the eigenvalue are "a" & "-d", hence it is a saddle point.
  - The physical meaning is that without the predator (y), the prey population is going to have a growth, while with absence of prey, the predator will have decay.
  - This also means the whole system will never extinct unless there is a sudden death of all the prey or predator
- For the stationary point (d/c, a/b), the Jacobian matrix is $\begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}$, the eigenvalues are $i\sqrt{ca}$ & $-i\sqrt{ca}$, hence it is a center. The point around is neither attracted nor repelled away.
  - The physical meaning is the whole population is going to oscillate around the means (value of stationary point).

<u>Another approach for stability</u>

- In 2-D system, for more complicated systems, eigenvalues might be difficult to compute, but if we are only interested in the signs of the eigenvalues, we could investigate the sign of the trace and the determinant instead. By definition, eigenvalues for a 2-D system are found this way.

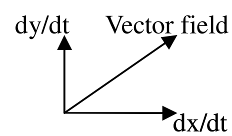
$$\text{For matrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \lambda = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12})}}{2}$$

- We could see that the sum of two eigenvalues is $(a_{11} + a_{22})$, and the product is $(a_{11}a_{22} - a_{21}a_{12})$

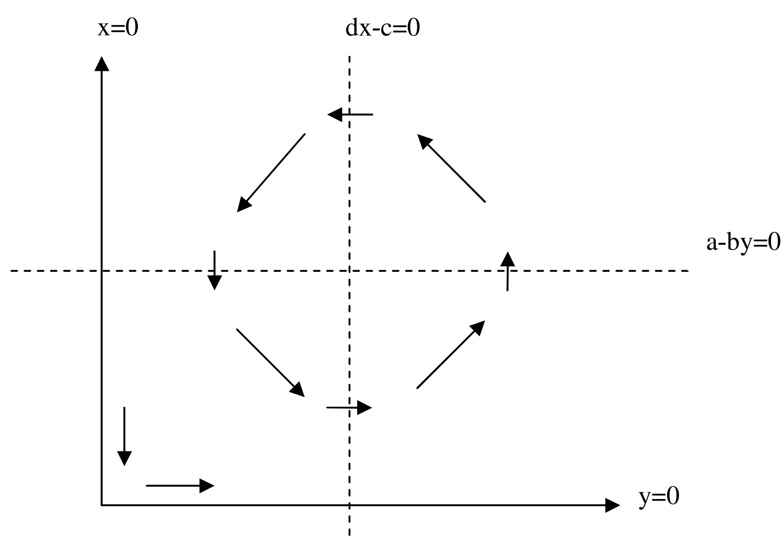
- Trace is defined as $(a_{11} + a_{22})$, and determinant is defined as $(a_{11}a_{22} - a_{21}a_{12})$
- Hence we are able to conclude the signs of two eigenvalues by looking at the trace and determinant only.
  - If both eigenvalues are negative (stable), then the trace (sum) must be negative, and the determinant (product) must be positive.

- In conclusion, the stability of a stationary point can be characterized as follow:
  - If the trace is negative AND the determinant is positive, it is stable
  - Otherwise it is unstable
  - Special case when the trace is zero, it is a center

<u>Vector Field Representation</u>

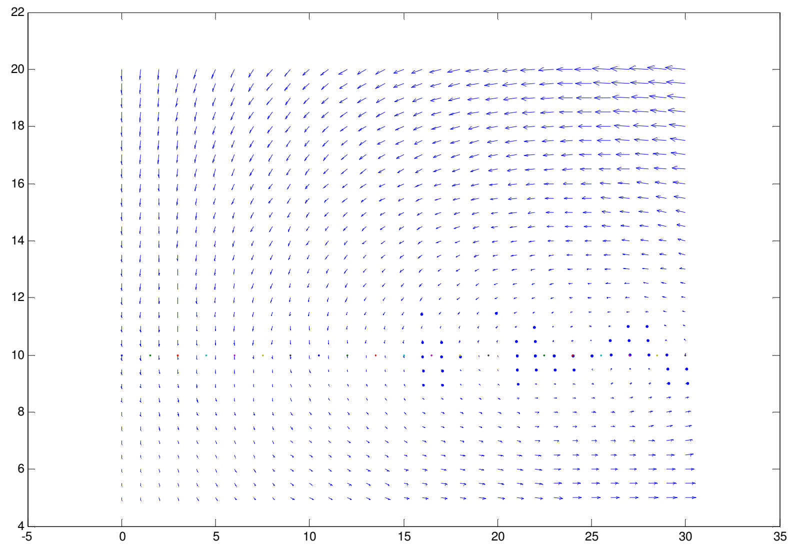


- Vector Field Representation is a powerful numerical method to analyze the change of a single point in the given plane.
- In our 2-D analysis of points on X-Y plane, the vector is the resultant vector of two vectors dx/dt and dy/dt in the x and y direction respectively.
- Hence, the vector at a point can be found by calculating dx/dt and dy/dt numerically at the point.
- With a set of the vectors, we could easily identify the general behaviours of the system.
- We are more interested in the behaviours of points around the stationary points.
- By using the sign of the dx/dt and dy/dt, we could identify the general trend of the vector fields.
- The general trend is easier to identify by group points into regions bounded by the dx/dt=0 line and the dy/dt=0 line (Nulclines).
- In the Lotka-Volterra system, dx/dt=0 when x=0 or a-by=0, dy/dt=0 when y=0 or –c+dx=0, through analysis of the region bounded by these nulclines (by calculating a few sampled points), we could define the general direction of the vector field.

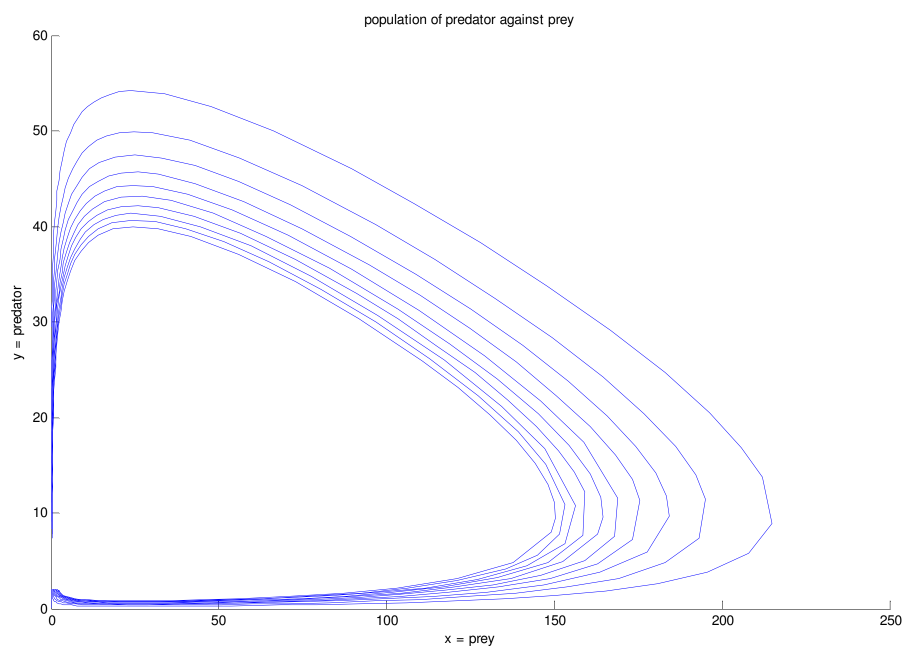


- For numerical analysis, a Matlab programme has been created to generate Vector Field plots with a given set of ODEs and parameters.

- Since through stability analysis, the nature of the stationary point is not dependant on any parameters. Hence we choose a set of parameters a = 1 b = 0.1 c = 0.02 d = 0.5, for illustration purpose. The following graph is plotted

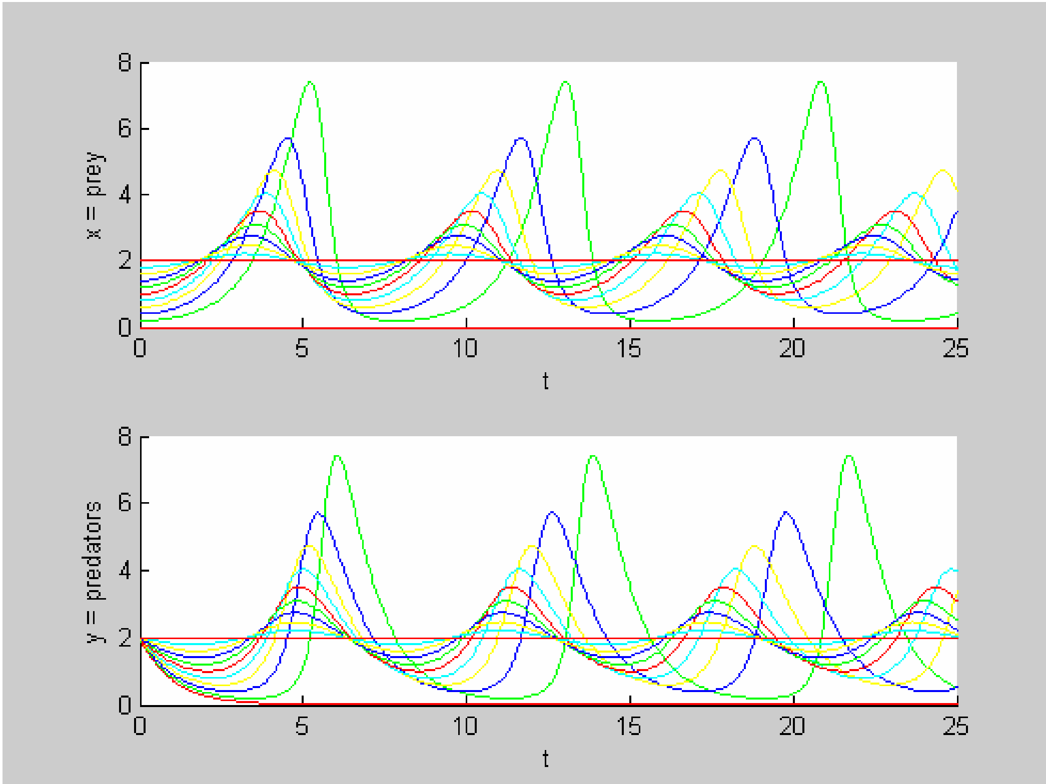


- The general trend of "center" characteristic stationary point is indeed the same as we predicted
- The contour of the Predator against Prey can also be plotted by a Matlab programme. The following graph, which uses the same values of parameter given above with the initial population of predator of 2, is generated by increasing the initial prey population from 0 to 2 by 0.2 each time.



- From the graph above, we could see that the contour is clustered together near the axis and the origin; this is due to the saddle nature of the stationary point (origin).

- If required, we could also draw in the time axis to describe the oscillation from each contour, the following is a sample of what we could generate.



Conclusion

- By using above analysis, we should be able to characterize a given dynamic system, and being able to tell more about the systems behaviours.
- Using Lotka-Volterra model we have analysed:
  - As we could see from the graph above, each contour is an enclosed encirclement, i.e. it is a periodic oscillation on its own.
  - Any perturbation can drive the oscillation into a different cycle.
  - It is especially chaotic if the perturbation is near the origin.
  - Hence we should set our second stationary point and initial condition far away from the axis and origin to avoid the drastic change in the output waveform.

Post stability analysis

- After the all the analysis, we might be able to suggest on how we can tune our system to our requirement.
- For instance, if we are using Lotka-Volterra as our model:
  - With a given set of possible parameters from experiments, we will use the analysis above and determine which sets of parameters best suit our interest.
  - With the proposed set of parameters and initial conditions, we will run a sensitivity test to characterize each parameter and the expected output.