Notes to Thursday October 9 Lecture

In lecture, we modeled a mass on a spring as

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_d / \mathrm{m}$$

where

$$\gamma = \frac{b}{2m} \quad \text{and} \quad \omega_o^2 = \frac{k}{m}$$

From Part (c) of PSet #5, you will find that the magnitude of the frequency response of the system is given by

$$\left|\hat{h}(\omega)\right| = \frac{Qm}{k}\left[Q^2(1-\frac{\omega^2}{\omega_o^2})^2 + \frac{\omega^2}{\omega_o^2}\right]^{-1/2} \qquad \text{Eq(1)}$$

where

$$Q = \frac{\omega_0}{2\gamma} = \frac{\omega_o m}{b}.$$

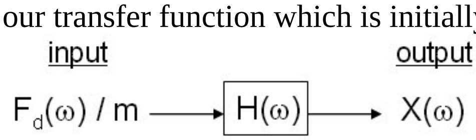
Note that h(t) is now defined according to the addendum that I sent out a few days ago:

$$\ddot{h} + 2\gamma\dot{h} + \omega_0^2 h = \delta(t)$$

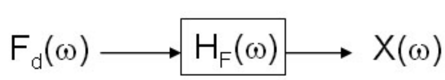
where h(t) is the impulse response. The output from the impulse acceleration is now given by f(t) = v_o·h(t).

To relate the frequency response to the Brownian motion of the cantilever, we let the drive force F_d arise from thermal energy k_bT. To do this, we imagine that the cantilever (or mass on spring) is being bombarded by particles at random times and directions. We can model the drive force that results from these random collisions as white noise

What we need to do now is solve for the magnitude of this white noise, S_F^o. This constant sets the lower limit of force measurement, since any applied force that is smaller than this will be drowned out by the Brownian motion. To do this, we first redefine our transfer function which is initially an acceleration



to a force



where

$$H_f(\omega) = \frac{1}{m}H(\omega) = \frac{1}{m}\hat{h}(\omega)$$

By taking the magnitude squared of relation between input and output of the new system gives

$$S_x(\omega) = S_F^o\left|H_F(\omega)\right|^2 \qquad \text{Eq(2)}$$

There are two more key equations that we need to find S_F^o:

i) Parsaval's theorem gives

$$\langle x^2 \rangle = \frac{1}{2\pi}\int_0^\infty S_x(\omega)d\omega \qquad \text{Eq(3)}$$

ii) Equipartition theorem gives

$$\frac{1}{2}k_bT = \frac{1}{2}k\langle x^2 \rangle \qquad \text{Eq(4)}$$

which when combined gives

$$k_bT/k = \langle x^2 \rangle$$

$$= \frac{1}{2\pi}\int_0^\infty S_F^o \frac{Q^2}{k^2}\left[Q^2(1-\frac{\omega^2}{\omega_o^2})^2 + \frac{\omega^2}{\omega_o^2}\right]^{-1} d\omega$$

or

$$S_F^o = \frac{4kk_bT}{Q\omega_o} \qquad \text{Eq(5)}$$

This expression can be used to find the minimum detectable force for a given bandwidth B.

There are two important aspects of this expression:

i) Lowering the spring constant improves the resolution at which force can be measured.

ii) If we plug in the definition of Q

$$Q = \frac{\omega_0}{2\gamma} = \frac{\omega_o m}{b}$$

we find that

$$S_F^o = 4k_bTb$$

which says that the minimum detectable force is simply related to the damping by the constant k_bT.

We can now plug in these expression into Eq(2) to find the position PSD

$$S_x(\omega) = \frac{4k_bT}{Qk\omega_o}\left[(1-\frac{\omega^2}{\omega_o^2})^2 + \frac{\omega^2}{Q^2\omega_o^2}\right]^{-1}$$

which can be used to find the minimum detectable position. In contrast to force PSD, increasing the spring constant will improve the resolution at which position can be measured.

A few more notes:

- S_x has units of position^2 per Hz even though ω is in rad/sec since we originally defined the PSD in Hertz (Sept 18 lecture). You will want to substitute ω with $2\pi f$ since Hz is used in the lab.
- p. 17 of the lab module doesn't use the same notation that that we used in lecture and the bottom equation is incorrect (ω should be replaced by *f* and *z* should be *x*). These lecture notes should be used instead.