Condensed structural solutions to the disturbance rejection and decoupling problems with stability

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In this paper, it is shown that a static state feedback control law rejects the disturbance and simultaneously performs input–output decoupling, while ensuring internal stability of the closed loop system, if and only if an integer equality holds for the global undisturbed system and the row subsystems of the combined plant (where the disturbance is handled as a control input), namely in terms of their respective total contents (sums of the orders of the multiplicities of its infinite and finite unstable invariant zeros).

1. Introduction

The aim of this paper is to provide new condensed structural conditions for the existence of a static state feedback control law performing input–output decoupling of a disturbed linear time-invariant system, while ensuring disturbance rejection and closed-loop internal stability.

The combined problem of disturbance rejection and input–output decoupling was first discussed in Chang and Rhodes (1975) and Fabian and Wonham (1975), more than twenty years ago. More recently, this problem has been revisited in Paraskevopoulos *et al.* (1991) using matrix tools, while the structural approach, through input–output tools, has been used in Dion *et al.* (1994) to obtain *easy-to-verify* solvability conditions (corresponding to our Lemma 7). Note also that contributions from Koussiouris and Tzierakis (1984, 1995) give a complementary algebraic and structural treatment for this combined problem.

As the main result of the research referenced above, a nice property has been found. Indeed, when no stability constraint is imposed, it has been established that the combined problem is solvable if and only if each problem, separately, has a solution. Recently, it has been established in Martínez García and Malabre (1995), using a geometric approach, that this separation property still holds when the internal stability of the closed-loop system is required. Thanks to this separation solvability, necessary and sufficient conditions for the existence of a solution to the combined problem were obtained in Malabre and Martínez García (1994) after some fusion of the existing structural results for the disturbance rejection problem with stability (Malabre and Martínez García 1993) and the row by row decoupling problem with stability (Martínez García and Malabre 1994). The structural solution given in Malabre and Martínez García (1994) is expressed in terms of *a couple of integer equalities* completely characterized by the parameters of the global undisturbed system and the row subsystems of the combined plant (where the disturbance is handled as a control input), namely in terms of their respective infinite and unstable contents (sums of the orders of the multiplicities of these singularities). In the present paper we go further: we reduce the two integer equalities to *only one*, namely in terms of the total content (sum of the infinite and unstable contents) of the undisturbed system and the total content of the row subsystems of the combined plant. To do this, we first obtain a new solution for the disturbance rejection problem with stability and for the row by row decoupling with stability (these new solutions are expressed in terms of total contents and not in terms of both infinite and unstable contents, as in Malabre and Martínez García (1993) and Martínez García and Malabre (1994). The separation property mentioned above is then used to solve the problem.

For the sake of brevity, we shall only consider here the case when the disturbance is available in the control law. If the disturbance is not available for the control law, various tricks exist to turn back to the previous case (see, for instance, Malabre and Martínez García 1995).

Even though most of the results given here could be naturally extended to systems which are just stabilizable, we have decided to restrict our exposition to systems which are controllable and right invertible.

First of all, we shall present some basic concepts (mainly the infinite, the unstable invariant and the total contents of a linear time-invariant system) and the new structural solvability conditions for both the disturbance rejection problem with stability and the (regular) row by row decoupling problem with stability.

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In the second part of the paper we shall present the new structural solution for the combined problem, using for this the separation property mentioned above and the new structural solutions of both disturbance rejection and decoupling. The structural solution amounts simply to comparing a couple of positive integers, namely the total content (the sum of the infinite and finite unstable invariant contents) of the undisturbed system and the total sum of the total contents of the row subsystems of the combined plant. A detailed example is then included to illustrate our result.

2. Basic concepts

First of all we shall introduce some standard notation.

The ith row of a matrix C is denoted by c_i. The identity map on an n-dimensional space is denoted by I_n. the field of complex numbers is denoted by $\mathbb{C}$. The open left-half complex plane is denoted by $\mathbb{C}_-$ and the Laplace variable is denoted by s.

The *normal rank* of a matrix M, i.e. the rank of M as a matrix with entries in the ring of polynomials (or in the field of rational functions), in s with constant coefficients, is denoted normal rank $[M]$. We shall only write rank$[M]$ if all the entries of M are taken in the field of complex numbers.

Given the linear time-invariant system (A, B, C) described by

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad t \geq 0 \\ y(t) &= Cx(t), \qquad t \geq 0 \end{aligned}\right\} \tag{1}$$

where $x(t)$ is the n-dimensional state, $u(t)$ is the m-dimensional control input and $y(t)$ is the p-dimensional output ($A: \mathcal{X} \to \mathcal{X}$, $B: \mathcal{U} \to \mathcal{X}$, and $C: \mathcal{X} \to \mathcal{Y}$ are linear maps represented in particular bases by real constant matrices), we have the following definitions.

2.1. *Structure at infinity*

From an algebraic point of view, the *structure at infinity* of any linear system (A, B, C) described by (1) or equivalently by its proper $(p \times m)$ transfer function matrix

$$T(s) := C(sI_n - A)^{-1}B$$

is described by the *multiplicity orders of its zeros at infinity*, since such systems have no poles at infinity (see, for instance, Vardulakis 1991). This structure can be derived from the so-called Smith–McMillan form at infinity of $T(s)$, say $\Delta_\infty(s)$, which is a canonical form under the following *biproper* transformation group

$$T(s) \to B_1(s)\,T(s)\,B_2(s)$$

where the $B_i(s)$'s are proper rational matrices, which are invertible and with proper inverse. Indeed

$$\Delta_\infty(s) = \begin{bmatrix} \begin{bmatrix} s^{-n_1} & & & \\ & s^{-n_2} & & \\ & & \ddots & \\ & & & s^{-n_r} \end{bmatrix} & 0_{r\times(m-r)} \\ 0_{(p-r)\times r} & 0_{(p-r)\times(m-r)} \end{bmatrix}$$

where the non-increasing list of positive integers $\{n_1, n_2, \ldots, n_r\}$ is the list of the *multiplicity orders of the zeros at infinity* of the system (A, B, C), with $r :=$ normal rank $[T(s)]$. From a geometric point of view, various equivalent definitions have been given for this structure, see for instance Commault and Dion (1981) for the original one and Malabre (1982) for other geometric characterizations.

2.2. *Finite invariant zeros*

The *system matrix* of (A, B, C) (see Rosenbrock 1972, Vardulakis 1991) is the polynomial matrix $SM_{(A,B,C)}(s)$ defined as

$$SM_{(A,B,C)}(s) := \begin{bmatrix} sI_n - A & B \\ -C & 0_{p\times m} \end{bmatrix}$$

The *invariant factors* not equal to one of $SM_{(A,B,C)}(s)$ are called the *non-trivial invariant polynomials* of (A, B, C) and their roots are called the ***finite** invariant zeros* of (A, B, C). Clearly, $z \in \mathbb{C}$ is a finite invariant zero of (A, B, C) if and only if

$$\text{rank}\,[SM_{(A,B,C)}(z)] < \text{normal rank}\,[SM_{(A,B,C)}(s)]$$

For a geometric characterization of the finite invariant zeros see for instance Wonham (1985) and Basile and Marro (1992).

Note that if (A, B, C) is a minimal state description, i.e. (A, B) controllable and (C, A) observable, the finite invariant zeros of system (A, B, C) coincide with the so-called transmission zeros of its transfer function matrix $T(s)$. Recall that $z \in \mathbb{C}$ is a transmission zero of $T(s)$ if and only if z is a root of a numerator of the Smith–McMillan form of $T(s)$ (see for instance Vardulakis 1991).

2.3. *The contents*

The *content at infinity* of (A, B, C), noted as $C_\infty(A, B, C)$, is the total sum of the orders of its zeros at infinity (see for instance, Verghese (1978) for the original definition and Malabre and Martínez García (1993) and Martínez García and Malabre (1994) for some geometric complements).

The *unstable* (*invariant*) *content*, noted as $C^+(A, B, C)$, is the total sum of the multiplicity orders of its unstable invariant zeros (see for instance Malabre and Martínez García (1993) and Martínez García and Malabre (1994) for some geometric complements).

The *total content*, noted as $C_\infty^+(A, B, C)$, is the sum of $C_\infty(A, B, C)$ and $C^+(A, B, C)$.

Note that $C_\infty(A, B, C)$, $C^+(A, B, C)$ and $C_\infty^+(A, B, C)$ are positive integers.

3. Problems statement

3.1. *Disturbance rejection*

Let us consider the disturbed time-invariant system (A, B, C, E) described by

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ed(t) \qquad t \geq 0 \\ y(t) &= Cx(t), \qquad\qquad\qquad\quad t \geq 0 \end{aligned}\right\} \quad (2)$$

where $d(t)$ is a q-dimensional disturbance, and the map E is such that $E : \mathcal{D} \to \mathcal{X}$. When $d(t)$ is considered as a 'fictitious' control input, and not as a disturbance, we shall denote $(A, [B\ E], C)$ the resulting combined system.

The *disturbance rejection problem* (DRP) is then defined as follows:

> 'Does there exist a control law $u(t) = Fx(t) + Hd(t)$, $t \geq 0$, where $F : \mathcal{X} \to \mathcal{U}$ and $G : \mathcal{D} \to \mathcal{U}$, such that the output $y(s)$ be independent on $d(s)$ in (2)?' Or equivalently
>
> 'Do there exist maps $F : \mathcal{X} \to \mathcal{U}$ and $H : \mathcal{D} \to \mathcal{U}$ such that
>
> $$C(sI_n - (A + BF))^{-1}(BH + E) \equiv 0?\text{'} \quad (3)$$

When we add the constraint $\sigma(A + BF) \in \mathbb{C}_-$ to DRP, where $\sigma(A + BF)$ denotes the spectrum of $A + BF$, we have the so called *disturbance rejection problem with stability* (DRPS).

Remark 1: An obvious necessary condition for $\sigma(A + BF) \in \mathbb{C}_-$ to hold is that (A, B) must be a stabilizable pair.

3.2. *Regular decoupling*

Let the controllable system (A, B, C) be given. Assuming that the number of inputs is greater than or equal to the number of outputs (i.e. $m \geq p$), the *row by row decoupling problem via regular static state feedback* (DP) is then defined as

> 'Do there exist maps $F : \mathcal{X} \to \mathcal{U}$ and $G : \mathcal{U} \to \mathcal{U}$, with G regular (invertible), such that
>
> $$C(sI_n - (A + BF))^{-1}BG \equiv [\text{diag}\{\zeta_1(s), \ldots, \zeta_p(s)\} \,|\, 0]$$
>
> where diag stands for diagonal matrix and each $\zeta_i(s)$ is a strictly proper (non-zero) rational transfer function matrix, with $v_i(s)$ as its input and $y_i(s)$ as its output?'

If such F and G, regular, exist, then we shall say that (A, B, C) is a *regularly row by row decouplable* system.

When we add the constraint $\sigma(A + BF) \in \mathbb{C}_-$ to DP we have the so-called (regular) *row by row decoupling problem with stability* (DPS).

Remark 2: An obvious necessary condition for $\sigma(A + BF) \in \mathbb{C}_-$ to hold is that (A, B) must be a stabilizable pair. In fact, a usual assumption within the DP is that (A, B) is controllable. Some geometric results (as in Martínez García and Malabre 1994, 1995 for instance) explicitly use this assumption. Note that the results presented there obviously remain true when (A, B) is just stabilizable. However, for sake of consistency, we shall assume from now that (A, B) is controllable. Right invertibility of (A, B, C) is also a well known necessary solvability condition for DP.

3.3. *Simultaneous disturbance rejection and regular decoupling*

The simultaneous disturbance rejection and (regular) row by row decoupling problem (DRDP) is defined as

> 'Given the system (A, B, C, E), find conditions for the existence of a static state feedback control law $u(t) = Fx(t) + Gv(t) + Hd(t)$, $t \geq 0$, with the constraint G regular, such that
>
> $$C(sI_n - (A + BF))^{-1}[BG \,|\, BH + E] = [\text{diag}\{\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_p(s)\} \,|\, 0 \,|\, 0]$$

where $\varepsilon_i(s)$, for all $i \in \{1, 2, \ldots, p\}$, are proper (non-zero) rational transfer functions.'

When we add the constraint $\sigma(A + BF) \in \mathbb{C}_-$ to DRDP we have the so-called *simultaneous disturbance rejection and (regular) row by row decoupling problem with stability* (DRDPS).

4. Main results

Let us now denote $T_{[B\ E]}(s)$ and $T_{[B]}(s)$ the transfer function matrices of the combined plant $(A, [B\ E], C)$ and the undisturbed system (A, B, C), respectively, i.e. $T_{[B\ E]}(s) := [T_{[B]}(s) \,|\, C(sI_n - A)^{-1}E]$ and $T_{[B]}(s) := C(sI_n - A)^{-1}B$.

4.1. *Structural solution of DRP and DRPS*

4.1.1. *Structural solution of DRP*

Lemma 1 (Theorem 4, Malabre and Martínez García 1993): *Given the disturbed system (A, B, C, E), the disturbance rejection problem is solvable if and only if* normal rank $[T_{[B]}(s)] =$ normal rank $[T_{[B\ E]}(s)]$ and $C_\infty(A, B, C) = C_\infty(A, [B\ E], C)$.

We can now recall the *structural solution of DRPS*, which has been obtained in Malabre and Martínez García (1993) through the extension of the results of Lemma 1 to the present stable case.

Lemma 2 (Corollary 1, Malabre and Martínez García 1993): *Given a disturbed system (A, B, C, E), under the assumption (A, B) controllable, the disturbance rejection problem with stability is solvable if and only if the undisturbed system (A, B, C) and the combined plant $(A, [B\ E], C)$ have the same normal rank, the same infinite content and the same unstable (invariant) content, i.e.:*

(a) normal rank $[T_{[B]}(s)]$ = normal rank $[T_{[B\ E]}(s)]$

(b) $C_\infty(A, B, C) = C_\infty(A, [B\ \ E], C)$

(c) $C^+(A, B, C) = C^+(A, [B\ \ E], C)$.

In fact, if (a) in Lemma 2 holds, we can further reduce the integer equalities (b) and (c) to only one, namely in terms of the total contents of both the undisturbed system (A, B, E) and the combined plant $(A, [B E] C)$. In order to do this we shall need the following properties.

Property 1: *For a disturbed system (A, B, C, E) under the assumption* normal rank $[T_{[B]}(s)]$ = normal rank $[T_{[B\ E]}(s)]$ *it is always true that*

$$C_\infty(A, B, C) \geq C_\infty(A, [B\ \ E], C)$$

Proof: Since normal rank $[T_{[B]}(s)]$ = normal rank $[T_{[B\ E]}(s)]$ both the undisturbed (A, B, C) and the combined plant $(A, [B\ E], C)$ have the same number of zeros at infinity. On the other hand, the presence of E cannot increase the multiplicity orders of the zeros at infinity. Thus, the result follows immediately. □

As far as the unstable content is concerned, we have similarly, the following property.

Property 2: *For a disturbed system (A, B, C, E), under the assumption* normal rank $[T_{[B]}(s)]$ = normal rank $[T_{[B\ E]}(s)]$, it is always true that

$$C^+(A, B, C) \geq C^+(A, [B\ \ E], C)$$

Proof: Let us first give a sketchy proof. In what follows we shall consider, without loss of generality, that the disturbed system is stable (if not, because of controllability of (A, B), we can always find a static state feedback which stabilizes the system).

Since normal rank $[T_{[B\ E]}(s)$ = normal rank $[T_{[B]}(s)]$, a complex frequency z such that rank $[T_{[B E]}(z)]$ is smaller than normal rank $[T_{[B E]}(s)]$ is also such that rank $[T_B(z)]$ is smaller than normal rank $[T_B(s)]$ (if not, normal rank $[T_B(s)]$ would be equal to rank $[T_B(z)]$, which is a contradiction). It then follows that a finite transmission zero of $T_{[B\ E]}(s)$ is also a finite transmission zero† of $T_{[B]}(s)$.

Because of stability, the finite unstable invariant zeros and the finite unstable transmission zeros of the stable system $(A, [B\ E], C)$ coincide, since no unstable pole-zero cancellations are possible in $T_{[B\ E]}(s)$, It then follows that all the finite unstable transmission zeros of the combined system $(A, [B\ E], C)$ are included in the set of the finite unstable transmission zeros of the undisturbed system (A, B, C), which coincide with the set of its finite unstable invariant zeros (because of stability). The previous arguments are quite simple but cannot directly apply to the case of multiple zeros. For that more general situation, classical results from completion problems actually give a complete answer. Indeed, it has been shown in Thompson (1979, Theorem 2) and Sá (1979) that in a principal ideal ring, a matrix P of normal rank k and invariant factors $h_1(P) | h_2(P) | h_3(P) | \cdots | h_k(P)$ (here $h_i(P) | h_{i+1}(P)$ means that the ith invariant factor $h_i(P)$ divides the $(i+1)$th factor $h_{i+1}(P)$) may be augmented (in the same principal ideal ring), with a single row to obtain a matrix Q, of normal rank k and invariant factors $h_1(Q) | h_2(Q) | h_3(Q) | \cdots | h_k(Q)$ if and only if

$$h_1(Q) | h_1(P) | h_2(Q) | h_2(P) \cdots | h_k(Q) | h_k(P)$$

As far as the disturbance rejection problem is concerned, the system matrix of $(A, [B\ E], C)$ can be seen as the augmentation of the system matrix of (A, B, C) with new columns (originated by the presence of the disturbances). These columns play in this case the role that the rows, which augment matrix P to obtain matrix Q, play in the result cited above. Since the rank condition in Property 2, the result of Thompson (1979, Theorem 2) and Sá (1979) can be directly applied to obtain a complete proof of Property 2. Indeed, from the fact that the largest invariant polynomial of the system matrix of the composite system divides the largest invariant polynomial of the system matrix of the undisturbed system immediately follows that all the zeros of $A, [B\ E], C)$ (unstable or not) are zeros of (A, B, C). This is *a fortiori* true for the unstable ones and consequently $C^+(A, [B\ E], C) \leq C^+(A, B, C)$, which ends the proof. □

We can now present Theorem 1.

Theorem 1: *Given a disturbed system (A, B, C, E), under the assumption (A, B) controllable, the disturbance rejection problem with stability is solvable if and only if both the undisturbed system (A, B, C) and the combined*

† Recall that a complex frequency z is called a finite transmission zero of a system $T(s)$ if z is a root of a numerator of the Smith–McMillan form of $T(s)$.

plant $(A, [B\ E], C)$ *have the same normal rank and the same total content, i.e.*

(a) normal rank $[T_{[B]}(s)]$ = normal rank $[T_{[B\ E]}(s)]$
(b) $C^+_\infty(A, B, C) = C^+_\infty(A, [B\ \ E], C)$.

Proof: If (b) and (c) in Lemma 2 hold, then

$$C^+_\infty(A, B, C) = C^+_\infty(A, [B\ \ E], C) \tag{4}$$

Conversely, suppose that (4) and (a) hold. We then have from Properties 1 and 2 that $C_\infty(A, B, C) = C_\infty(A, [B\ E], C)$ and $C^+(A, B, C) = C^+(A, [B\ E], C)$, which ends the proof. □

Note that (a) in Theorem 1 always holds if (A, B, C) is right invertible, since right invertibility of (A, B, C) implies right invertibility of $(A, [B\ E], C)$. We then have the following corollary.

Corollary 1: *Assuming that* (A, B, C) *is right invertible and* (A, B) *is controllable, the disturbance rejection problem with stability is solvable if and only if* (A, B, C) *and* $(A, [B\ E], C)$ *have the same total content, i.e.* $C^+_\infty(A, B, C) = C^+_\infty(A, [B\ E], C)$.

4.2. *Structural solutions of DP and DPS*

4.2.1. *Structural solution of DP.* The next lemma has been proved in Martínez García and Malabre (1994). The proof, which is not presented here, uses the fact that for any right invertible system (A, B, C), with m inputs and p outputs, $p \le m$, the outputs can always be numbered in order that $n_i' \le n_i$, for all $i \in \{1, \ldots, p\}$, where n_i' denotes the order of the zero at infinity of the row-subsystem (A, B, c_i).

Let us recall that we restrict our attention here to (A, B, C) systems which are right invertible and controllable.

Lemma 3 (Dion and Commault 1993)**:** *The row by row decoupling problem is solvable if and only if* $C_\infty(A, B, C) = \sum_{i=1}^{p} C_\infty(A, B, c_i)$.

Remark 3: The original form of this result has been given in Descusse and Dion (1982) in terms of an equality between both sets $\{n_i\}$ and $\{n_i'\}$. In fact a closer look at their geometric proof shows that the authors indeed establish the result of Lemma 3, which is an equivalent condition, due to the natural ordering relation existing between $\{n_i\}$ and $\{n_i'\}$ and the full row rank assumption.

4.2.2. *Structural solutions of DPS*. We can recall the following *structural* result, which is given in Martínez García and Malabre (1994)

Lemma 4 (Theorem 6, Martínez García and Malabre (1994)**:** *Assuming* (A, B, C) *right invertible and* (A, B) *controllable, the* (*regular*) *row by row decoupling problem with stability is solvable if and only if*

(a) $C_\infty(A, B, C) = \sum_{i=1}^{p} C_\infty(A, B, c_i)$

(b) $C^+(A, B, C) = \sum_{i=1}^{p} C^+(A, B, c_i)$

In fact, the two solvability conditions in Lemma 4 can be reduced to only one. For this purpose we shall consider the following properties of a system (A, B, C).

Property 3: *For a system* (A, B, C), *supposed to be right invertible,* $C_\infty(A, B, C) \ge \sum_{i=1}^{p} C_\infty(A, B, c_i)$ *always holds.*

Proof: This a classical result, quite direct from Verghese (1978) (see for instance Dion *et al.* 1994). □

Property 4: *For any right invertible system* (A, B, C), $C^+(A, B, C) \ge \sum_{i=1}^{p} C^+(A, B, c_i)$.

Proof: We again just give a sketchy proof, as for Property 2. A complete proof has already been proposed in Koussiouris (1984) within an algebraic setting (see also Icart *et al.* 1990 for a geometric counterpart).

In what follows we shall consider, without loss of generality, that (A, B, C) is stable (if not, because of our assumptions, we can always find a static state feedback which stabilizes the system).

Let us denote $T_i(s)$ the transfer function of the row subsystem (A, B, c_i), i.e. $T_i(s) = c_i(sI_n - A)^{-1}B$. Then, the transfer function matrix of (A, B, C) is given by

$$C(sI_n - A)^{-1}B := \begin{bmatrix} c_1(sI_n - A)^{-1}B \\ c_2(sI_n - A)^{-1}B \\ \vdots \\ c_p(sI_n - A)^{-1}B \end{bmatrix}$$

Since its right invertibility, it is clear that this matrix has a loss of rank (for a given complex frequency) if one of its row subsystems is equal to zero. It means that a finite transmission zero of a row subsystem (A, B, c_i), $i \in \{1, 2, \ldots, p\}$, is also a finite transmission zero of the global system (A, B, C).

Because of the impossibility of unstable pole-zero cancellations to occur in an internally stable system, the set of the finite unstable invariant zeros of the global system (A, B, C) coincide with the set of its finite unstable transmission zeros (the sample applies for the row subsystems). As a finite transmission zero of a row subsystem is also a finite transmission zero of the global system, we have that $\sum_{i=1}^{p} C^+(A, B, c_i) \le C^+(A, B, C)$, which ends the proof. □

We can now present the following synthetic version of Lemma 4.

Theorem 2: *Assuming* (A, B, C) *right invertible and* (A, B) *controllable, the row by row decoupling problem with stability is solvable if and only if*

$$C^+_\infty(A, B, C) = \sum_{i=1}^{p} C^+_\infty(A, B, c_i)$$

Proof: Note that if (a) and (b) in Lemma 4 hold, then

$$\begin{aligned} C^+_\infty(A, B, C) &:= C_\infty(A, B, C) + C^+(A, B, C) \\ &= \sum_{i=1}^{p} C_\infty(A, B, c_i) + \sum_{i=1}^{p} C^+(A, B, c_i) \\ &=: \sum_{i=1}^{n} C^+_\infty(A, B, c_i) \end{aligned}$$

Conversely, if (A, B, C) is right invertible, we have from Properties 3 and 4 that the positive integer equality $C^+_\infty(A, B, C) = \sum_{i=1}^{p} C^+_\infty(A, B, C_i)$ implies $C_\infty(A, B, C) = \sum_{i=1}^{p} C_\infty(A, B, c_i)$ and $C^+(A, B, C) = \sum_{i=1}^{p} C^+(A, B, c_i)$. □

Let us now consider simultaneous disturbance rejection and decoupling with stability.

4.3. *Structural solution of DRDP and DRDPS*

First of all let us recall that:

Lemma 5 (Chang and Rhodes 1975): *The simultaneous disturbance rejection and (regular) row by row decoupling problem (DRDP) has a solution if and only if both the disturbance rejection problem (DRP) and the regular row by row decoupling problem (DP) are solvable separately.*

And for the stable case:

Lemma 6 (Theorem 9, Martínez García and Malabre 1995): *The simultaneous disturbance rejection and (regular) row by row decoupling problem with stability (DRDPS) has a solution if and only if both the disturbance rejection problem with stability (DRPS) and the regular row by row decoupling problem with stability (DPS) are solvable separately.*

Lemma 5 has been used in Dion *et al.* (1994) and Malabre and Martínez García (1994) to obtain the following structural solution of the simultaneous disturbance rejection and (regular) row by row decoupling problem, when no internal stability of the closed-loop system is required.

4.3.1. *Structural solution of DRDP.*

Lemma 7 (Dion *et al.* 1994, Malabre and Martínez García 1994): *Assuming* (A, B, C) *right invertible and* (A, B) *controllable, the simultaneous disturbance rejection and (regular) row by row decoupling problem is solvable if and only if* $C_\infty(A, B, C) = \sum_{i=1}^{p} C_\infty(A, [B\ E], c_i)$.

Thanks to Lemma 6 and in the light of the new structural solutions of both the disturbance rejection (Corollary 1, since the right invertibility of the system is a necessary condition for decoupling to have a solution) and the (regular) row by row decoupling problems with stability (Theorem 2), we can now present our final result.

4.3.2. *Structural solution of DRDPS.*

Theorem 3: *Given the disturbed system* (A, B, C, E), *under the assumptions* (A, B, C) *right invertible and* (A, B) *controllable, the simultaneous disturbance rejection and (regular) row by row decoupling problem with stability is solvable if and only if*

$$C^+_\infty(A, B, C) = \sum_{i=1}^{p} C^+_\infty(A, [B\ E], c_i) \tag{5}$$

Proof: From Lemma 6, Corollary 1 and Theorem 2, DRDPS is solvable if and only if

$$\left.\begin{aligned} C^+_\infty(A, B, C) &= C^+_\infty(A, [B\ E], C) \\ C^+_\infty(A, B, C) &= \sum_{i=1}^{p} C^+_\infty(A, B, c_i) \end{aligned}\right\} \tag{6}$$

We have to prove that (5) and (6) are equivalent. For this purpose, note that Properties 1 and 2 imply, under our assumptions (right invertibility of (A, B, C) implies normal rank $[T_{[B]}(s)] =$ normal rank $[T_{[B\ E]}(s)]$), that the positive integer inequalities

$$\left.\begin{aligned} C^+_\infty(A, B, C) &\geq C^+_\infty(A, [B\ E], C) \\ \sum_{i=1}^{p} C^+_\infty(A, B, c_i) &\geq \sum_{i=1}^{p} C^+_\infty(A, [B\ E], c_i) \end{aligned}\right\} \tag{7}$$

always hold.

On the other hand, right invertibility of system (A, B, C) and controllability of (A, B) imply, from Properties 3 and 4, that the following inequalities always hold

$$\left.\begin{aligned} C^+_\infty(A, B, C) &\geq \sum_{i=1}^{p} C^+_\infty(A, B, c_i) \\ C^+_\infty(A, [B\ E], C) &\geq \sum_{i=1}^{p} C^+_\infty(A, [B\ E], c_i) \end{aligned}\right\} \tag{8}$$

Thus

$$C^+_\infty(A, B, C) \geq C^+_\infty(A, [B \quad E], C) \geq \sum_{i=1}^{p} C^+_\infty(A, [B \quad E], c_i) \qquad (9)$$

and

$$C^+_\infty(A, B, C) \geq \sum_{i=1}^{p} C^+_\infty(A, B, c_i) \geq \sum_{i=1}^{p} C^+_\infty(A, [B \quad E], c_i) \qquad (10)$$

It is then clear that if (5) holds, then from (9) and (10) we have that

$$C^+_\infty(A, B, C) = C^+_\infty(A, [B \quad E], C) = \sum_{i=1}^{p} C^+_\infty(A, B, c_i)$$

For the reverse part, suppose now that there exists a control law $u(t) = Fx(t) + Gv(t) + Hd(t)$, $t \geq 0$, such that

$$C(sI_n - A_F)^{-1}[BG | BH + E] = [\mathrm{diag}\{\zeta_1(s), \ldots, \zeta_p(s)\} | 0 | 0]$$

and

$$\sigma(A_F) \subset \mathbb{C}_-$$

where $A_F := A + BF$ and with $\zeta_i(s)$ standing for a strictly proper transfer function matrix. Since $u(t) = Fx(t) + Gv(t) + Hd(t)$, $t \geq 0$, rejects the disturbance and performs input–ouput decoupling with internal stability of the closed-loop system, it means that $C^+_\infty(A, B, C) = C^+_\infty(A, [B \; E], C)$ and $C^+_\infty(A, B, C) = \sum_{i=1}^{p} C^+_\infty(A, B, c_i)$. Then defining

$$F_c := \begin{bmatrix} F \\ 0 \end{bmatrix} \quad G_c := \left[\begin{array}{c|c} G & H \\ 0 & I_q \end{array}\right] \quad A_{Fc} := A + [B \quad E] F_c$$

we have that

$$C(sI_n - A_{Fc})^{-1}[B \quad E] G_c = C(sI_n - A_F)^{-1}[BG | BH + E] = [\mathrm{diag}\{\zeta_1(s), \ldots, \zeta_p(s)\} | 0 | 0]$$

and $\sigma(A_{Fc}) = \sigma(A_F) \subset \mathbb{C}_-$. This means that $(A, [B \; E], C)$ is also regularly row by row decouplable with stability. Consequently $C^+_\infty(A, [B \; E], C) = \sum_{i=1}^{p} C^+_\infty(A, [B \; E], c_i)$, which ends the proof. □

5. An illustrative example

Let us consider the stable system (A, B, C, E) with

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -5 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} \alpha \\ 1 \\ \beta \\ \gamma \end{bmatrix}$$

Has the simultaneous disturbance rejection and (regular) row by row decoupling problem a stable solution for these data?

First of all, note that (A, B) is a controllable pair (in fact the realization (A, B, C) is minimal). Now, because of the stability of the open-loop system, we can obtain directly all the structural information that we need, in order to conclude about the solvability of our current problem, from an input–output approach. In particular, the information concerning the unstable finite invariant zeros is present in the transfer function matrices related with the system, since no unstable pole/zero cancellations are possible. The unstable finite invariant zeros coincide with the unstable finite transmission zeros.

We proceed then to the obtention of the transfer functions matrices $T_B(s) := C(sI_4 - A)^{-1}B$ and $T_E(s) := C(sI_4 - A)^{-1}E$

$$T_B(s) = N_B(s)\begin{bmatrix} \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{(s+1)^2} \end{bmatrix} \quad \text{and}$$

$$T_E(s) = N_E(s)\frac{1}{(s+1)^2}$$

where

$$N_B(s) = \begin{bmatrix} s-4 & 0 \\ s+1 & s+2 \end{bmatrix} \quad \text{and}$$

$$N_E(s) = \begin{bmatrix} s(1-5\alpha) - (4+5\alpha) \\ s(1+\beta+\gamma) + (1+\beta+2\gamma) \end{bmatrix}$$

5.1. *First case:* $\alpha = 0$

Let $T_{[B \; E]}(s)$ be the transfer function matrix of the combined system $(A, [B \; E], C)$, i.e.

$$T_{[B \; E]}(s) = [T_B(s) \quad T_E(s)] = \frac{1}{(s+1)^2}[N_B(s) \quad N_E(s)]$$

In order to obtain the structural information that we need to conclude about the solvability of our current problem, we proceed as follows.

5.1.1. *Unstable finite invariant zero structure.* The transmission zeros of (A, B, C) are the roots of the invariant polynomials of $N_B(s)$, which are $\{1, (s-4)(s+2)\}$. It means that $s = 4$ and $s = -2$ are the transmission zeros of (A, B, C) and consequently $s = 4$ is the only unstable finite invariant zero that (A, B, C) has.

As far as the combined system $(A, [B\ E], C)$ is concerned, the only unstable finite invariant zero which is present is also $s = 4$, for any γ and for any β. Note that $s = -2$ is also a transmission zero of $(A, [B\ E], C)$, if $\gamma = 1$ and $\beta = 0$.

Now, for the row subsystems of (A, B, C) and $(A, [B\ E], C)$ we have the following situation.

Both (A, B, c_1) and $(A, [B\ E], c_1)$ have $s = 4$ as their only unstable finite invariant zero. On the other hand, no unstable finite invariant zeros are present in (A, B, c_2) and $(A, [B\ E], c_2)$.

5.1.2. *Structure of zeros at infinity.* Since

$$CB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is invertible, it immediately follows that the global structure at infinity of (A, B, C) is $\{1, 1\}$ as well as the row by row structure at infinity, i.e. $C_\infty(A, B, C) = 2$, $C_\infty(A, B, c_1) = 1$ and $C_\infty(A, B, c_2) := 1$. It is then also obvious that $C_\infty(A, [B\ E], C) = 2$, $C_\infty(A, [B\ E], c_1) = 1$ and $C_\infty(A, [B\ E], c_2) := 1$.

5.1.3. *Total contents.* Consequently, the total content $C_\infty^+(A, B, C)$ of (A, B, C) is given by

$$C_\infty^+(A, B, C) = C_\infty(A, B, C) + C^+(A, B, C) = 2 + 1 = 3$$

and as far as the total sum of the total contents of the row subsystems $\sum_{i=1}^{2} C_\infty^+(A, [B\ E], c_i)$ of $(A, [B\ E], C)$ is concerned

$$\sum_{i=1}^{2} C_\infty^+(A, [B\ \ E], c_i) = \sum_{i=1}^{2} (C_\infty(A, [B\ E], c_i) + C^+(A, [B\ E), c_i)) = 2 + 1 = 3.$$

Then $C_\infty^+(A, B, C) = 3 = \sum_{i=1}^{2} C_\infty^+(A, [B\ E], c_i)$, and because of Theorem 3, there exists at least one control law $u(t) = Fx(t) + Gv(t) + Hd(t)$, $t \geq 0$, which simultaneously rejects the disturbance and decouples the system under the constraint of internal stability.

As a particular static state feedback control law which rejects the disturbance and simultaneously decouples the system (while ensuring internal stability of the closed-loop system), for our current example (for the case $\alpha = 0$), we have

$$u(t) = Fx(t) + Hd(t) + Gv(t), \qquad t \geq 0$$

with

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad H = \begin{bmatrix} -1 \\ -\beta - \gamma \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Indeed

$$C(sI_4 - (A + BF))^{-1}[BG \mid BH + E] = \begin{bmatrix} \dfrac{s-4}{(s+1)^2} & 0 & 0 \\ 0 & \dfrac{1}{s+1} & 0 \end{bmatrix}$$

5.2. *Second case:* $\alpha \neq 0$

In this case, the information concerning the total content of (A, B, C) is not modified, i.e. $C_\infty^+(A, B, C) = 3$.

Now, we can easily check that

$$C_\infty^+(A, [B\ \ E], c_1) = 1$$

(we have a zero at infinity of order one, but we have no unstable finite invariant zero) and

$$C_\infty^+(A, [B\ \ E], c_2) = 1$$

Consequently:

$$C_\infty^+(A, B, C) = 3 \neq 2 = \sum_{i=1}^{2} C_\infty^+(A, [B\ \ E], c_i)$$

which implies that the simultaneous disturbance rejection and (regular) row by row decoupling problem has no stable solution when $\alpha \neq 0$. Let us now quickly show where this unsolvability comes from. We can easily check that

$$C_\infty^+(A, B, C) = 3 = \sum_{i=1}^{2} C_\infty^+(A, B, c_i)$$

which is a necessary and sufficient condition for the (regular) row by row decoupling problem to have a stable solution (see Theorem 2), but

$$C_\infty^+(A, B, C) = 3 \neq 2 = C_\infty^+(A, [B\ \ E], C) \qquad (11)$$

Indeed, the invariant polynomials of $[N_B(s)\ \ N_E(s)]$ are now the elements of the set $\{1, 1\}$ and consequently $(A, [B\ E], C)$ has no unstable finite invariant zero. On the other hand, the structure at infinity of $(A, [B\ E], C)$ is not affected by the choice of α, thus $C_\infty^+(A, [B\ E], C) = 2$.

Because of Theorem 1, equation (11) implies that the disturbance rejection problem has no stable solution for

$\alpha \neq 0$. Thus, we cannot solve the simultaneous disturbance rejection and (regular) row by row decoupling problem with stability (see Lemma 5).

Since $C_\infty(A, B, C) = 2 = C_\infty(A, [B\ E], C)$, we can always find a static state feedback control law which rejects the disturbance, but because of $C^+(A, B, C) = 1 \neq 0 = C^+(A, [B\ E], C)$ we cannot find a stable solution. This pathology (due to $\alpha \neq 0$) can be illustrated as follows.

Let $u(t) = \phi x(t) + \Gamma d(t)$, $t \geq 0$, denote any solution to the DRP. Then the precompensator $K(s)$ which is input–output equivalent to (ϕ, Γ), i.e. such that

$$K(s) = [I_2 - \phi(sI_4 - A)^{-1}B]^{-1}(\Gamma + \phi(sI_4 - A)^{-1}E)$$

satisfies $T_B(s)K(s) = -T_E(s)$.

For our simple example, such a solution $K(s)$ is unique

$$K(s) = [T_B(s)]^{-1}T_E(s)$$

$$= \begin{bmatrix} \dfrac{s(1-5\alpha)-(4+5\alpha)}{s-4} \\ \dfrac{(s+1)[s(1-5\alpha)-(4+5\alpha)]}{(s+2)(s-4)} + \dfrac{s(1+\beta+\gamma)+(1+\beta+2\gamma)}{s+2} \end{bmatrix}$$

Note that $K(s)$ is stable† only if $\alpha = 0$, which confirms our structural result. □

6. Conclusion

In this paper we have presented necessary and sufficient conditions for the solvability of the simultaneous disturbance rejection and decoupling problem by static state feedback with stability. The if and only if condition turns out to be simply the equality of two positive integers, i.e. the total content of system (A, B, C) and the sum of the total contents of the row subsystems of $(A, [B\ E], C)$.

In order to reduce the complexity of solvability analysis, it is not only of theoretical interest to obtain as compact as possible solvability conditions, it could be also of practical interest, see for example the case of structured systems (Dion *et al.* 1994) where the sum of the infinite zero orders (content at infinity) is much more easily available on the associated graph than the whole infinite structure. It also must be pointed out that the structural condition that we presented here is tool-independent, which is the main characteristic of the results obtained by the so-called structural approach.

† Which is necessary in order to guarantee the internal stability of the compensated system.

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References

Basile, G., and Marro, M., 1992, *Controlled Invariants and Conditioned Invariants in Linear System Theory* (New Jersey: Prentice-Hall).

Chang, M., and Rhodes, I. B., 1975, Disturbance localization in linear systems with simultaneous decoupling, pole assignment, or stabilization. *IEEE Transactions on Automatic Control*, **AC-20**, 518–523.

Commault, C., and Dion, J. M, 1981, Structure at infinity of linear multivariable systems: a geometric approach. *IEEE 20th Conference on Decision and Control*, San Diego CA, USA.

Descusse, J., and Dion, J. M., 1982, On the structure at infinity of linear square decoupled systems. *IEEE Transactions on Automatic Control*, **AC-27**, 971–974.

Dion, J. M., and Commault, C., 1993, Feedback decoupling of structured systems. *IEEE Transactions on Automatic Control*, **AC-38**, 1132–1135.

Dion, J. M., Commault, C., and Montoya, J., 1994, Simultaneous decoupling and disturbance rejection: a structural approach. *International Journal of Control*, **59**, 1325–1344.

Fabian, E., and Wonham, W. M., 1975, Decoupling and disturbance rejection. *IEEE Transactions on Automatic Control*, **AC-20**, 399–401.

Icart, S., Lafay, J. F., and Malabre, M., 1990, A unified study of the fixed modes of systems decoupled via regular static state feedback. *Joint Conference on New Trends in System Theory*, gênes-Italie, pp. 425–432 (Boston: Birkhäuser).

Koussiouris, T., 1984, A frequency domain approach to the block decoupling problem II: Pole assignment while block decoupling a minimal system by state feedback and a constant non singular input transformation and the observability of the block decoupled system. *International Journal of Control*, **32**, 443–464.

Koussiouris, T. G., and Tzierakis, K. G., 1984, Frequency-domain conditions for disturbance rejection and decoupling with stability or pole placement. *Automatica*, **32**, 229–234.

Koussiouris, T. G., and Tzierakis, K. G., 1995, Rejection of measurable disturbances with decoupling or pole placement. *Third European Control Conference (ECC'95)*, Rome, Italy, pp. 2557–2562.

Malabre, M., 1982, Structure à l'infini des triplets invariants. Applications à la poursuite parfaite de modèle. *Lecture Notes in Control and Information Sciences*, Vol. 44, pp. 43–53 (New York: Springer Verlag)

Malabre, M., and Martínez García, J. C., 1993, The modified disturbance rejection problem with stability: a structural approach. *Second European Control Conference (ECC'93)*, Groningen, The Netherlands, pp. 119–124.

Malabre, M., and Martínez García, J. C., 1994, Simultaneous disturbance rejection and decoupling with stability: a structural approach. *IEEE 33rd Conference on*

Decision and Control, Orlando, Florida, USA. December, pp. 885–886.

Malabre, M., and Martínez García, J. C.,1995, The partial model matching or partial disturbance rejection problem: geometric and structural solutions. *IEEE Transactions on Automatic Control*, **AC-40**, 356–360.

Martínez García, J. C., and Malabre, M., 1994, The row by row decoupling problem with stability: a structural approach. *IEEE Transactions on Automatic Control*, **AC-39**, 2457–2460.

Martínez García, J. C., and Malabre, M, 1995, Simultaneous disturbance rejection and row by row decoupling with stability: a geometric approach. *IEEE Transactions on Automatic Control*, **AC-40**, 365–369.

Paraskevopoulos, P. N., Koumboulis, F. N., and Tzierakis, K. G., 1991, Disturbance rejection with simultaneous decoupling of linear time invariant systems. *First European Control Conference (ECC'91)*, Grenoble, France.

Rosenbrock, H. H., 1972, The zeros of a system. *International Journal of Control*, **18**, 297–299 (and correction **20**, 525–527, 1974).

Marques de Sá. E., 1979, Imbedding conditions for λ-matrices. *Linear Algebra and Its Applications*, **24**, 33-50.

Thompson, R. C., 1979, Interlacing inequalities for invariant factors. *Linear Algebra and Its Applications*, **24**, 1–31.

Vardulakis, A. I. G., 1991, *Linear Multivariable Control: Algebraic Analysis and Synthesis Methods* (New York: Wiley).

Verghese, C. G., 1978, Infinite frequency behaviour in generalized dynamical systems. PhD Thesis, Stanford University, CT, USA.

Wonham, M., 1985, *Linear Multivariable Control: A Geometric Approach* (New York: Springer Verlag).