John Sy
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Modeling in Biology

Assignment 3

Question 1: A numerical exploration of the Lorenz system

We first consider the three dimensional Lorenz system shown below.

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - xz - y$$

$$\dot{z} = xy - bz$$

To get a good understanding of the system, we can start by finding the fixed points of the system by setting all of the equations to zero such that $\dot{x} = \dot{y} = \dot{z} = 0$. We obtain that the fixed points are:

$$x* = (0, 0, 0)\ and\ \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$$

From the form of the fixed points, it can be seen that (0,0,0) is always a fixed point, but the other fixed point only appears when $r > 1$. When $r < 1$, the fixed points are imaginary and are not seen on the phase plane.

We can also calculate the Jacobian of the system to obtain the stability analysis of the system. The Jacobian is as follows:

$$Jacobian_{(x,y,z)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

From this point, we can calculate the eigenvalues of the system via Matlab to determine the stability of all the fixed points. In the table below, we change the value of r and determine the eigenvalues of the jacobian for both the (0,0,0) fixed point and the other two fixed points. In the following analysis we fix the values of σ and b:

$$\sigma = 10, b = 8/3$$

r = 10	Fixed Point	Eigenvalues
	0 0 0	5.4659, -16.4659, -2.6667
	4.8990 4.8990 9.0000	-12.4757, -0.5955 - 6.1742i, -0.5955 + 6.1742i
	-4.8990 -4.8990 9.0000	-12.4757, -0.5955 - 6.1742i, -0.5955 + 6.1742i

For r = 10, we can see that (0,0,0) is a saddle node and the other fixed points are considered attracting spirals since two of the eigenvalues are imaginary. Thus, the two fixed points other than (0,0,0) will be the attractors of the system since it can be seen that the real component of the imaginary eigenvalues are negative. This negative real component signifies that the spirals are inward towards the fixed point.

r = 24.5	Fixed Point	Eigenvalues
	0 0 0	10.7865, -21.7865, -2.6667
	7.9162 7.9162 23.5000	-13.6523, -0.0072 - 9.5814i, -0.0072 + 9.5814i
	-7.9162 -7.9162 23.5000	-13.6523, -0.0072 - 9.5814i, -0.0072 + 9.5814i

For *r* = 24.5, the situation is the same as for *r* = 10 with (0,0,0) being a saddle node and the other fixed points being attracting spirals. Again, the two fixed points other than (0,0,0) will be the attractors of the system.

r = 25	Fixed Point	Eigenvalues
	0 0 0	10.9393, -21.9393, -2.6667
	8 8 24	-13.6825, 0.0079 - 9.6721i, 0.0079 + 9.6721i
	-8 -8 24	-13.6825, 0.0079 - 9.6721i, 0.0079 + 9.6721i

For *r* = 25, we have a slightly different case than for *r* = 10 or 24.5. Here we again see that (0,0,0) is a saddle node, but the other two fixed points have become repelling spirals as the real component of the eigenvalues have become positive meaning that they exponentially grow away from the fixed point. Later when we take a look at the phase plane and the bifurcation, we can see that this value of *r* is greater than the Hopf bifurcation point. Furthermore, since our fixed points have become unstable, there are no longer any long-term attractors of the system after the Hopf bifurcation.

r = 45	Fixed Point	Eigenvalues
	0 0 0	-2.6667, 16.1852, -27.1852
	10.8321, 10.8321, 44.0000	-14.6165, 0.4749 -12.6619i, 0.4749 +12.6619i
	-10.8321, -10.8321, 44.0000	-14.6165, 0.4749 -12.6619i, 0.4749 +12.6619i

r = 220	Fixed Point	Eigenvalues
	0 0 0	-2.6667, 41.6195, -52.6195
	24.1661 24.1661 219.0000	-17.2826, 1.8080 -25.9337i, 1.8080 +25.9337i
	-24.1661 -24.1661 219.0000	-17.2826, 1.8080 -25.9337i, 1.8080 +25.9337i

For *r* = 45 and 220, we have a similar situation for when *r* = 25. Since we see the positive real value in the eigenvalue corresponding to two fixed points which are not (0,0,0), they are no longer stable fixed points. Again, for these values of *r*, there are no long-term attractors of the system.

We can see a definite change of behavior between *r* values of 24.5 and 25. This Hopf bifurcation occurs at the r value:

$$r = r_H = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right)$$

For our specific values of b and σ, we find that $r_H = 24.736$. In the phase plane analysis, we can see this change of behavior occurring in figure 1. We can see that for *r* = 10 and 24.5, we get a spiral getting attracted to the fixed points (which are the long-term attractors of the system). For *r* > 24.5, we see that there are no longer any long-term attractors, but the behavior is chaotic and does not seem to follow a patter except that the trajectories seem to remain in a bounded area around unstable fixed points. For r = 10 and 24.5, it can be seen that the trajectories depend on the initial conditions and will either go to the positive or negative fixed point (figures 2a and 2b).

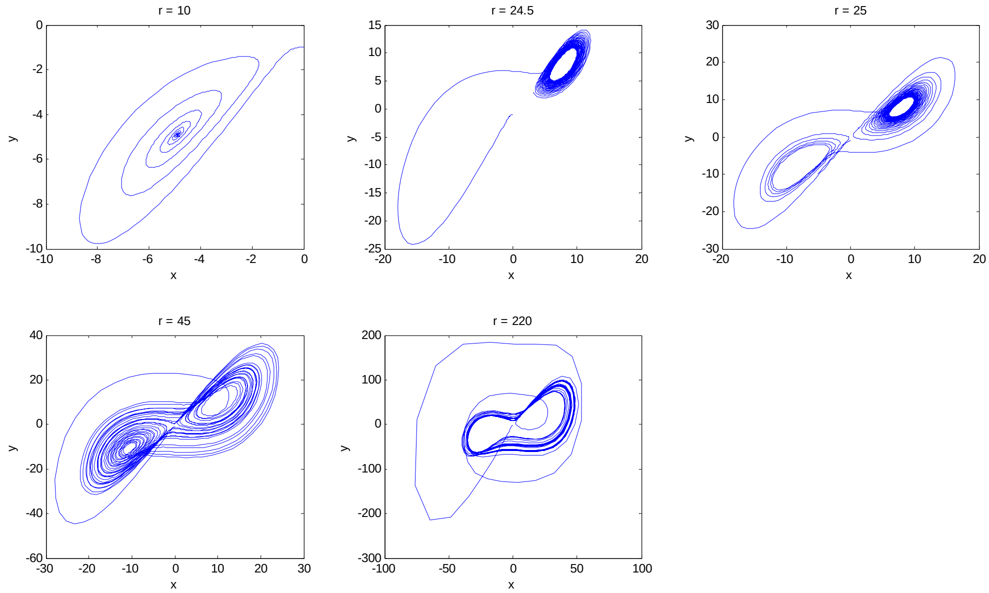


Figure 1. Phase plane x vs. y for various r values

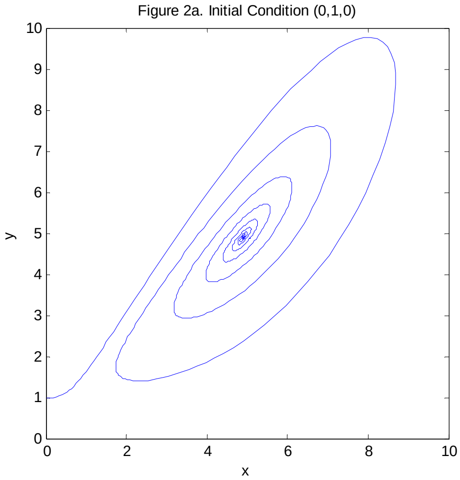


Figure 2a. Initial Condition (0,1,0)

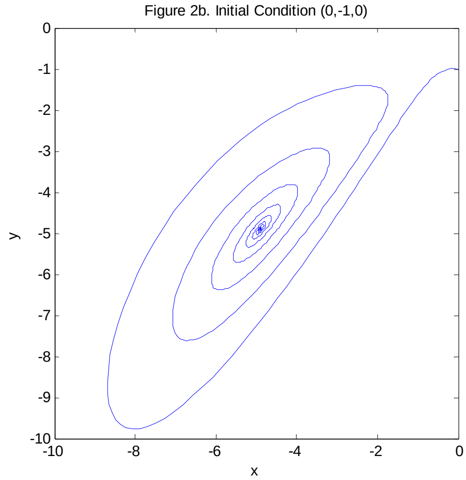


Figure 2b. Initial Condition (0,-1,0)

We can further visualize the chaotic nature of the system by plotting the time evolution as shown in figure 3. Chaotic behavior can be seen starting with r = 25. From our numerical analysis above, we know that for r = 24.5, the system will continue to be periodic and we predict that the long-term behavior will be similar to what is occurring already. For r > 24.5, we see a chaotic behavior which is aperiodic but bounded between values, two key elements a chaotic system has.

Another key element a chaotic system has is sensitivity to initial conditions. This is demonstrated by taking two initial values which are very close together.

$$I_1 = (8.8756, 16.1229, 11.5828)$$

$$I_2 = (8.8757, 16.1230, 11.5829)$$

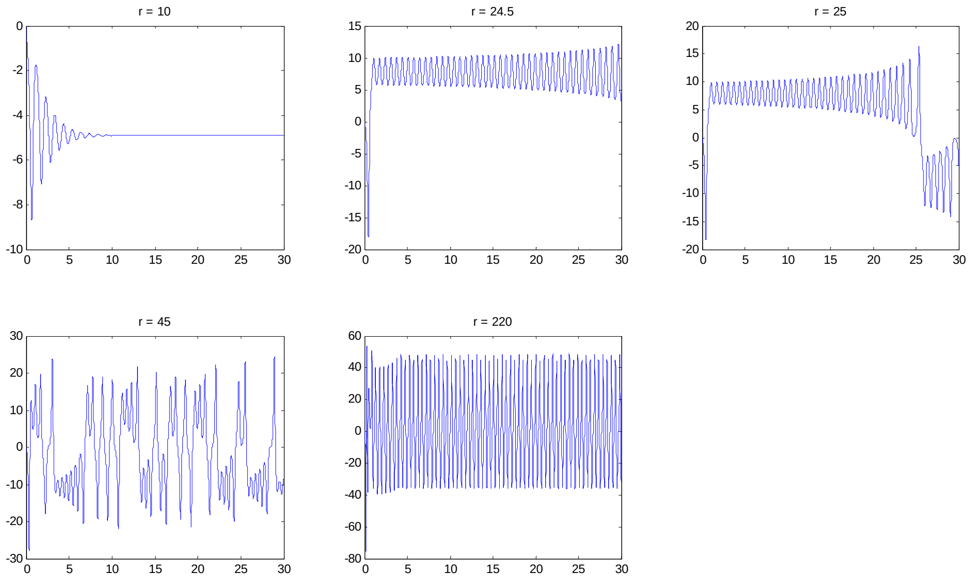
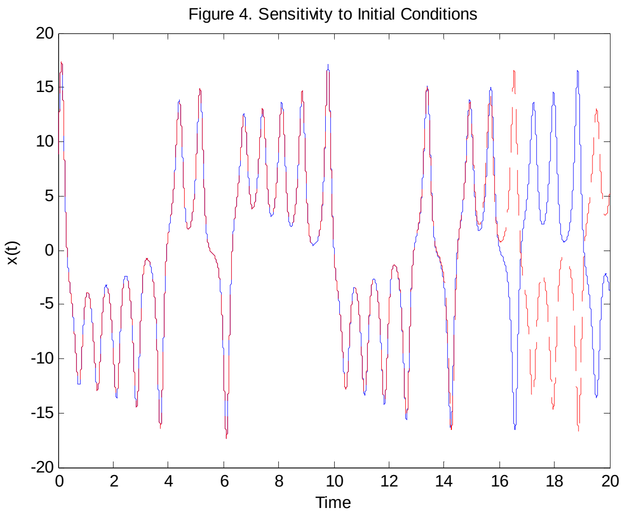


Figure 3. Time evolution of the Lorenz system

The plots of the two initial conditions are shown superimposed in figure 4 (blue corresponding to initial conditions I_1 and red dashed line corresponding to initial conditions I_2).

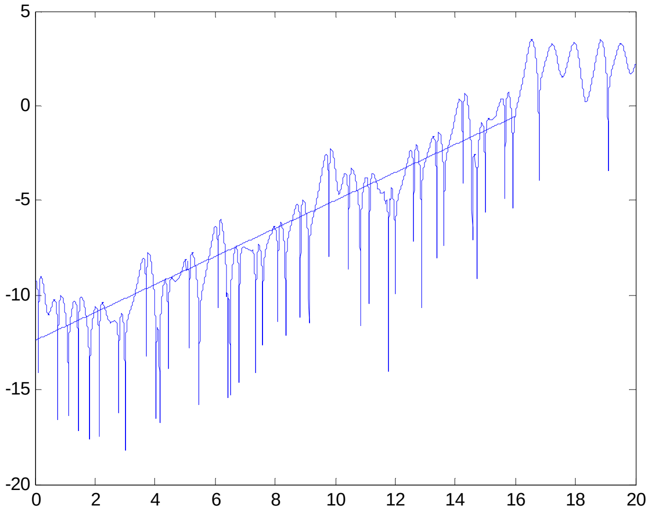


We can see that at approximately when time = 15, the two trajectories begin to diverge significantly. We can calculate the difference to calculate the Lyapunov exponent of the system which is derived from knowing that as time approaches 0, the two trajectories will be very close. As time increases, the two trajectories begin to diverge and the distance between the two is bounded by the elliptical volume in which the trajectories are contained. The Lyapunov exponent, λ, gives the rate of exponential divergence of the two trajectories and gives the limit to which we can predict the behavior of the system. By considering two neighboring trajectories, numerical studies have shown that the following relationship holds true.

$$\|\delta(t)\| \sim \|\delta_0\| e^{\lambda t}$$

Plotting $ln\|\delta_0\|$ vs t (figure 5), we obtain a curve that is close to a straight line with a positive slope λ. A straight line is then fit to the curve to find the value of the slope. The significance of the positive slope is that the trajectories are always diverging. If the slope were to be negative, then the trajectories would be considered converging and the system cannot be considered chaotic. In our system, λ was found to be equal 0.74. The fact that it is positive correlates with the idea that our system is indeed chaotic.

Figure 5. Finding the Lyapunov Exponent



Question 2: Period doubling and chaos in an oscillatory system