Part 3: Steady Points of the System

1) Equations of the Steady Points

The steady points of the Dimension-less version of the system

$$\begin{cases} \dfrac{dU}{ds} = \dfrac{U}{1+U} - \dfrac{BUV}{B_0+U} - EU \\ \dfrac{dV}{ds} = \dfrac{CUV}{1+UV} - DV \end{cases}$$

are given by the positive solutions $(U \geq 0, V \geq 0)$ of the system

$$\begin{cases} (1) \Leftrightarrow 0 = \dfrac{U}{1+U} - \dfrac{BUV}{B_0+U} - EU \\ (2) \Leftrightarrow 0 = \dfrac{CUV}{1+UV} - DV \end{cases}$$

Equation (1) yields $U = 0$ or $0 = \dfrac{1}{1+U} - \dfrac{BV}{B_0+U} - E$

Equation (2) yields $V = 0$ or $\dfrac{CU}{1+UV} = D \Leftrightarrow V = \dfrac{C}{D} - \dfrac{1}{U}$

2) Simple Steady Points

Plugging $V = 0$ into equation (1) yields 2 steady points

$S_1(0,0)$ and $S_2\left(\dfrac{1-E}{E}, 0\right)$

Admissibility

The steady point $S_1(0,0)$ is admissible whatever the values of the parameters

Conversely $S_2\left(\dfrac{1-E}{E}, 0\right)$ is admissible if $E \leq 1$. For E=1 (before it disappears) it merges with $S_1(0,0)$.

3) Other Steady Points

The condition $U = 0$ is incompatible with $V = \frac{C}{D} - \frac{1}{U}$. However $0 = \frac{1}{1+U} - \frac{BV}{B_0 + U} - E$ is.

Before studying the condition we set as with the previous dynamic system $R = C/D$.

The system of equations $(\otimes) \Leftrightarrow \begin{cases} BV = \frac{B_0 + U}{1+U} - E(B_0 + U) \\ V = R - \frac{1}{U} \end{cases}$ yield the other steady-points of the system. We will prove that we can get between 1 and 3 other points.

We eliminate the ordinate V to obtain the equation in U:

$$(\Delta) \Leftrightarrow \qquad B\left(R - \frac{1}{U}\right) = \frac{B_0 + U}{1+U} - E(B_0 + U)$$

$$(\Delta) \Leftrightarrow \qquad (BR - 1 + EB_0) - \frac{B}{U} + \frac{(1 - B_0)}{1+U} + EU = 0$$

$$(\Delta) \Leftrightarrow \qquad (BR - 1 + EB_0)U(1+U) - B(1+U) + (1 - B_0)U + EU^2(1+U) = 0$$

$$(\Delta) \Leftrightarrow \qquad EU^3 + (BR - 1 + EB_0 + E)U^2 + (B(R-1) + (E-1)B_0)U - B = 0$$

The case E=0 has been studied before. We can now assume $E > 0$.
The coordinate U of the other steady points are therefore also roots of the cubic

$$P(U) = U^3 + \alpha U^2 + \beta U - \delta = 0$$

where $\begin{cases} \alpha = \left(B\frac{R}{E} + B_0 + \frac{E-1}{E}\right) \\ \beta = (B(R-1) + (E-1)B_0)/E \\ \delta = \frac{B}{E} \end{cases}$

Since $\delta>0$ we are sure to have at least one positive root for P(U) – and we have at most 3. We are also sure that 0 cannot be a root. The condition for the existence of more than one positive roots are complex and have been studied in details in the scans available with this presentation. The results we have obtained a re a variation on the classic results by Cardano (see document on Cardano's formulae) stating that a cubic has

- one real root only if its discriminant Δ is strictly positive
- two real roots (one a double) if its discriminant is 0
- three real roots only if its discriminant is strictly negative

Admissibility Conditions

For the system to yield admissible steady points, the solutions U of the cubic P(U) must be positive. But they must also verify $0 \leq R - \frac{1}{U}$ or equivalently $0 \leq \frac{1}{1+U} - E$. The conditions can be simplified into $U \geq 1/R$ or equivalently $U \leq \frac{1-E}{E}$.

The conditions imply <u>R and E be linked</u>: $\frac{1-E}{E} \geq \frac{1}{R}$ that is

$$E \leq \frac{R}{R+1} \text{ or equivalently } R \leq \frac{E}{1-E}$$

It can be proven (see scans for more details) that $P(U) \geq 0 \;\; if \;\; U \geq \frac{1-E}{E}$ for admissible combinations of R and E. Therefore the positives roots of the cubic P(U) all satisfy $U \leq \frac{1-E}{E}$ and yield admissible solutions.

4) Summary for the other Steady Points

Existence of Other Steady Points

The remaining Steady Points of the Dynamic System are obtained by finding the positive roots of a cubic P(U) that depends on 4 parameters only (B, B_0, E, R) if E and R satisfy:

$$E \leq \frac{R}{R+1} \text{ or equivalently } R \leq \frac{E}{1-E}$$

Every positive root U of P yields a Steady Points of the Dynamic System $\left(U, R-\frac{1}{U}\right)$.

Cubic P(U)

The cubic P(U) is : $P(U) = U^3 + \alpha U^2 + \beta U - \delta = 0$

where $\begin{cases} \alpha = \left(B\frac{R}{E} + B_0 + \frac{E-1}{E}\right) \\ \beta = \left(B(R-1) + (E-1)B_0\right)/E \\ \delta = \frac{B}{E} \end{cases}$

We call Δ the discriminant of the cubic $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$

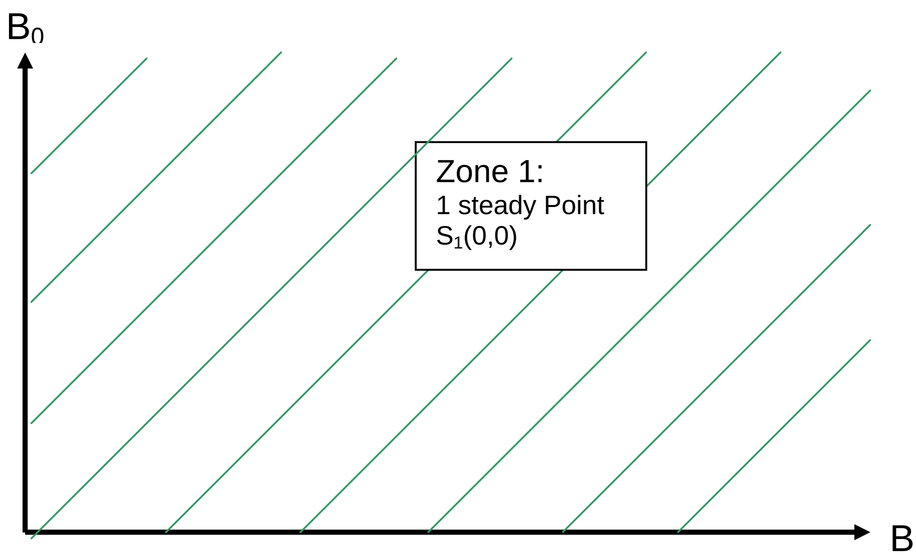
Where the coefficients p and q are $\begin{cases} p = \beta - \frac{\alpha^2}{3} \\ q = \frac{2\alpha^3}{27} - \frac{\alpha\beta}{3} - \delta \end{cases}$

P(U) has at least one positive root r_1. Depending on the combination of parameters P(U) may have two positive roots (r_1 r_2) or three (r_1 r_2, r_3). One can prove that

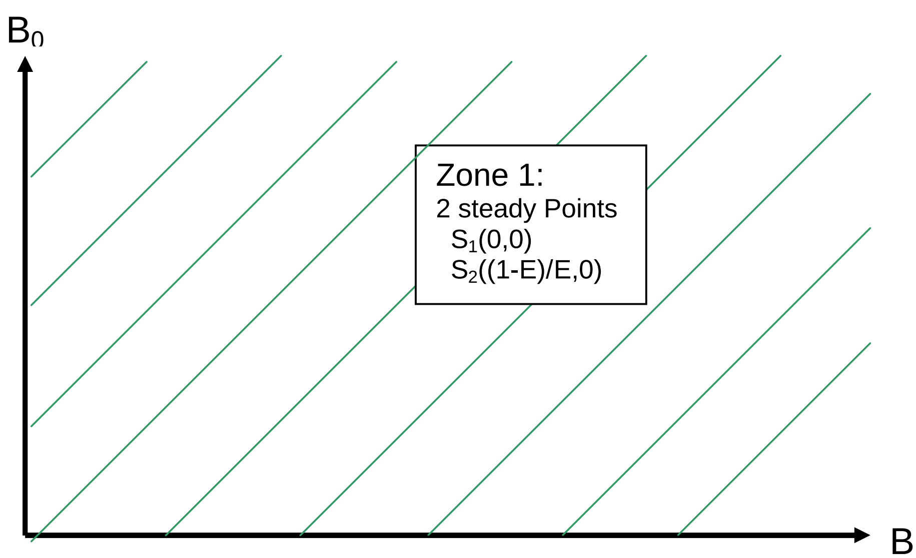
- the cubic has two real positive roots (one a double root) if $\begin{cases} \Delta = 0 \\ \alpha < 0 \\ \beta < \alpha^2/3 \end{cases}$
- the cubic has three real positive roots (one a double root) if $\begin{cases} \Delta < 0 \\ \alpha < 0 \\ \beta < \alpha^2/3 \end{cases}$
- else the cubic only has one positive root

5) Bifurcation Diagrams

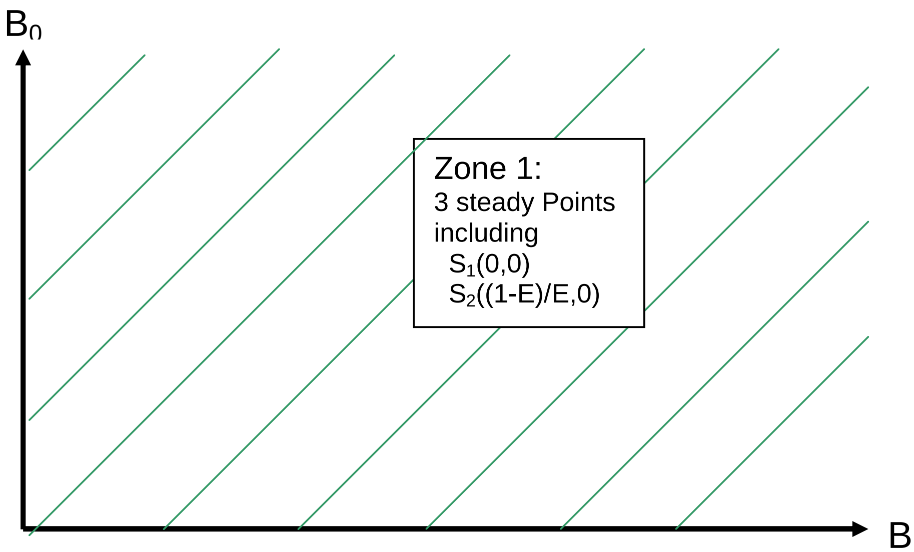
First Case $E \geq 1$



Second Case: $1 > E \geq R/(R+1)$



Case 3-A: E <R/(R+1) and R≤1



Case 3-B: E <R/(R+1) and R>1