Part 5: Stability Analysis of the System

We use the Dimension-less version of the system

$$\begin{cases} (1) \Leftrightarrow \dfrac{dU}{ds} = \dfrac{U}{1+U} - \dfrac{BUV}{B_0+U} - EU \\ (2) \Leftrightarrow \dfrac{dV}{ds} = \dfrac{CUV}{1+UV} - DV \end{cases}$$

The Jacobian of the system is

$$J(U,V) = \begin{pmatrix} \dfrac{1}{(1+U)^2} - \dfrac{BB_0V}{(B_0+U)^2} - E & -\dfrac{BU}{B_0+U} \\ \dfrac{CV}{(1+UV)^2} & \dfrac{CU}{(1+UV)^2} - D \end{pmatrix}$$

1) Stability of Steady Points S_1 and S_2

The Jacobian at the point S_1 is

$$J(0,0) = \begin{pmatrix} 1-E & 0 \\ 0 & -D \end{pmatrix}$$

Therefore

S_1 is an unstable saddle point if E<1
S_1 is stable if E>1
S_1 is somewhat stable if E=1

NB: The larger the E and D the more stable the point as we did expect.

The Jacobian at the point S_2 is

$$J(S_2) = \begin{pmatrix} \dfrac{1}{(1+U)^2} - E & -\dfrac{BU}{B_0+U} \\ 0 & CU - D \end{pmatrix}$$

that is

$$J(S_2) = \begin{pmatrix} E^2 - E & -\dfrac{BU}{B_0+U} \\ 0 & D\left(\dfrac{1-E}{E}R - 1\right) \end{pmatrix}$$

The first Eigenvalue of the matrix is $E^2 - E$, the second is $D\left(\frac{1-E}{E}R - 1\right)$.

The first Eigenvalue is negative if E<1 and null for E=1. The point S_2 does not exist if E>1.

The second Eigenvalue is negative if $R < \frac{E}{1-E}$ ie $\frac{R}{R+1} < E \leq 1$, else it is positive.

Therefore

> S_2 is an unstable saddle point if $E < \frac{R}{R+1}$
>
> S_2 is stable if $\frac{R}{R+1}E < 1$
>
> S_2 is somewhat stable if $E = \frac{R}{R+1}$

NB: Again the larger the E and D the more stable the point as we did expect.

2) Jacobian At the Other Points

We are going to modify the expression of the Jacobian of the system is

$$J(U,V) = \begin{pmatrix} \frac{1}{(1+U)^2} - \frac{BB_0V}{(B_0+U)^2} - E & -\frac{BU}{B_0+U} \\ \frac{CV}{(1+UV)^2} & \frac{CU}{(1+UV)^2} - D \end{pmatrix}$$

The coordinates of the other Steady Points are linked by the system of equations

$$(\otimes) \Leftrightarrow \begin{cases} BV = \frac{B_0+U}{1+U} - E(B_0+U) \\ V = R - \frac{1}{U} \end{cases}$$

It follows that $\frac{1}{1+UV} = \frac{D}{CU}$ and $\begin{cases} \frac{CU}{(1+UV)^2} - D = \frac{D^2}{CU} - D = \frac{D^2}{C}\left(\frac{1}{U} - R\right) = -\frac{D^2}{C}V \\ \frac{CV}{(1+UV)^2} = \frac{D^2V}{CU^2} \end{cases}$

$$J_{11} = \frac{1}{(1+U)^2} - \frac{BB_0V}{(B_0+U)^2} - E = \frac{1}{(1+U)^2} - \frac{B_0}{(B_0+U)^2}\left(\frac{B_0+U}{1+U} - E(B_0+U)\right) - E$$

that is

$$J_{11} = \frac{1}{(1+U)^2} - \frac{B_0}{(B_0+U)}\left(\frac{1}{1+U} - E\right) - E = \frac{1}{(1+U)^2}\left(1 - \frac{B_0(1+U)}{(B_0+U)}\right) + E\left(\frac{B_0}{(B_0+U)} - 1\right)$$

and finally

$$J_{11} = \frac{U}{(1+U)^2}\left(\frac{1-B_0}{(B_0+U)}\right) - E\frac{U}{(B_0+U)} = \frac{U}{(B_0+U)}\left(\frac{1-B_0}{(1+U)^2} - E\right)$$

Eventually we get the friendlier form of the Jacobian

$$J(U,V) = \begin{pmatrix} \frac{U}{(B_0+U)}\left(\frac{1-B_0}{(1+U)^2} - E\right) & -\frac{BU}{B_0+U} \\ \frac{D^2V}{CU^2} & -\frac{D^2V}{C} \end{pmatrix}$$

The Jacobian is very complex. We therefore split the stability study in two parts:

- Study of the Sign of the Determinant
- Study of the sign of the Trace

We restrict these studies to the case $E < R/(R+1)$. Indeed for the case $E = R/(R+1)$, the steady point(s) coincide with the point S_2 – the study for such case has therefore been done already (one eigenvalue is negative, one is zero)

3) Study of the Determinant for E<R/(R+1)

$$Det(J) = \frac{D^2UV}{C(B_0+U)}\left(\frac{B}{U^2} - \left(\frac{1-B_0}{(1+U)^2} - E\right)\right)$$

Det (J) has the same sign as $\frac{B}{U^2} + E - \left(\frac{1-B_0}{(1+U)^2}\right)$, Therefore

Det (J) has same sign as $Q(U) = EU^4 + 2EU^3 + U^2(E + B + B_0 - 1) + 2BU + B$

The Study of the sign of the expression $Q(U)$ is a fastidious affair if done by brute force. We were however, fortunate to find some simplifications. We ended up with a very powerful result

For the steady points of the system obtained by finding the roots of the cubic

$$P(U) = U^3 + \alpha U^2 + \beta U - \delta = 0$$

where
$$\begin{cases} \alpha = \left(B\dfrac{R}{E} + B_0 + \dfrac{E-1}{E} \right) \\ \beta = \left(B(R-1) + (E-1)B_0 \right)/E \\ \delta = \dfrac{B}{E} \end{cases}$$

The sign of determinant of the Jacobian J of the system is the same as the derivative of P at that steady point:

$$Sgn(Det(J(U,V))) = Sgn\left(P'(U) = 3U^2 + 2\alpha U + \beta\right)$$

And we can derive the following practical results

Determinant of the Jacobian

First Case: P(U) has one Positive Root U_s Only $Det(J(U_s,V_s)) > 0$

Second Case: P(U) has Two Positive Roots (the simple root U_s and the double root U_d)

For the simple root U_s $Det(J(U_s,V_s)) > 0$

for the double root U_d $Det(J(U_d,V_d)) = 0$

Last Case: P(U) has Three Positive Roots ($U_{s1} < U_{s2} < U_{s3}$)

For the middle root U_{s2} $Det(J(U_{s1},V_{s1})) < 0$: Saddle Point

For the other roots U_{s1} and U_{s3} $Det(J(U_s,V_s)) > 0$

Note: Again the regularizing effect of the dissipative term –EU can be felt to some extent. In the case E=0 the 'other' steady points were given by the roots of a quadratic and we had the possibility of a saddle point (Det (J)<0) that in effect would push the trajectories to infinity.
Here the 'other' steady points are given by a cubic. We still have the possibility of a saddle point, but this time the saddle point is 'controlled' by two points instead of one. More importantly as it was proven in Part 4, the trajectories remain bounded.

4) Study of the Trace for E<R/(R+1)

The trace of the Jacobian is $Tr(J) = \frac{U}{(B_0 + U)}\left(\frac{1-B_0}{(1+U)^2} - E\right) - \frac{DV}{R}$

We obtain a similar result on the sign of the trace of the Jacobian as we did with the dynamic system with E=0. The results are simpler to write in (B, B_0, R, D) coordinates.

> If $1 - B_0 < E(1+U)^2$ then Tr(J) is strictly negative regardless of D
>
> Conversely $1 - B_0 \geq E(1+U)^2$ the sign of the Trace changes as D increases
>
> If $D < D_{\lim}$ then Tr(J) is strictly positive
>
> If $D = D_{\lim}$ then Tr(J) =0
>
> If $D > D_{\lim}$ then Tr(J) <0
>
> where the limit value of D is $D_{\lim} = \frac{RU}{V(B_0 + U)}\left(\frac{1-B_0}{(1+U)^2} - E\right)$

NB: $1 - B_0 > E(1+U)^2$ is a more stringent condition than the previous $1 - B_0 > 0$. Likewise the expression of $D_{\lim}$ varies from the previous case by the introduction of the term –E.
Both are consequences of the dissipative term –EU which controls the flow and makes it more stable as E increases.

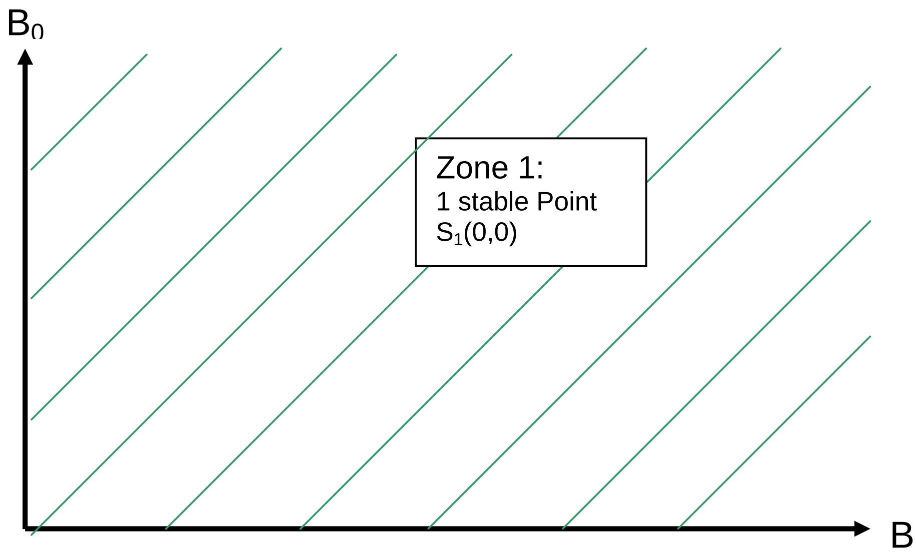
Unfortunately I was unable to find a simple , useful form of the condition $1 - B_0 > E(1+U)^2$. In the following bifurcation diagrams the shape of the zone of the (B, B_0) plane is just for the purpose of illustration. Truth is I have no idea what it looks like and it is likely to be made up of several connex regions. Furthermore for a choice of (B, B_0) that yields two or three roots for P(U) we need to check the condition for both points where Det (J)≥0 – it is very likely that for some points of the (B, B_0) plane the condition can be met for the smallest root of P(U) but not for the largest root.

Given that $\frac{1}{R} \leq U \leq \frac{1-E}{E}$, we get $1 - \frac{1}{E} \leq 1 - E(1+U)^2 \leq 1 - E\left(\frac{R+1}{R}\right)^2$

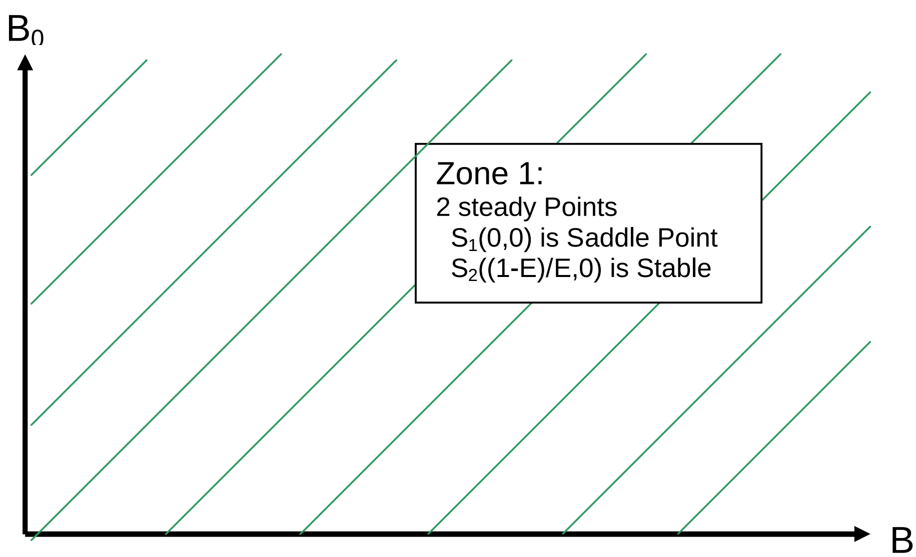
> We therefore have a necessary condition on B_0:
>
> $$B_0 \leq 1 - E\left(\frac{R+1}{R}\right)^2$$
>
> It follows that for $\frac{R}{R+1} > E > \left(\frac{R}{R+1}\right)^2$, all the ‘other’ steady points are stable

5) Bifurcation Diagrams

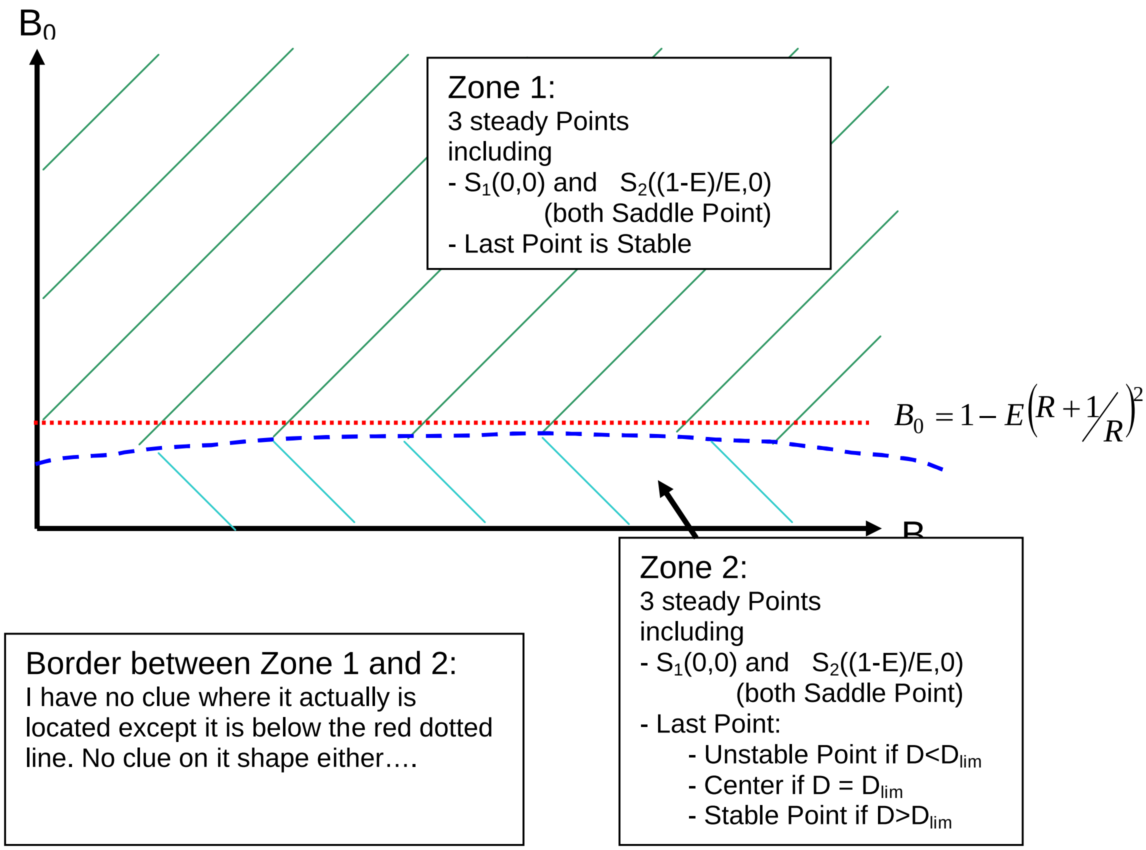
First Case E≥ 1



Second Case: 1> E ≥R/(R+1)

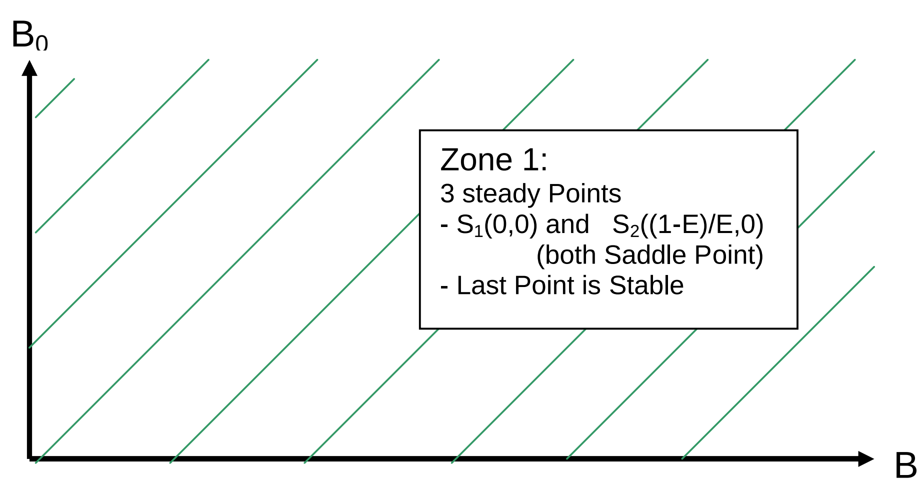


Case 3-1: E $\leq R^2/(R+1)^2$ and $R \leq 1$



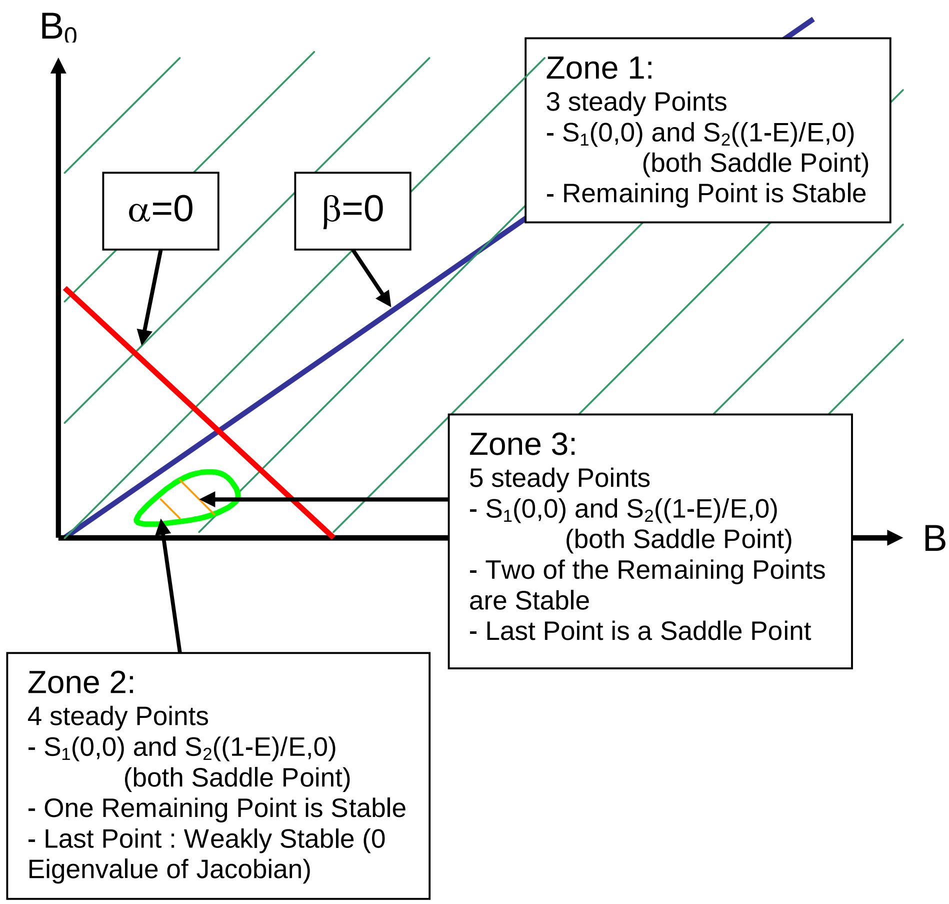
Case 3-2: E $> R^2/(R+1)^2$ and $R \leq 1$

The condition $E > R^2/(R+1)^2$ ensures that the trace of the Jacobian of the steady points linked to the polynomial P is negative.

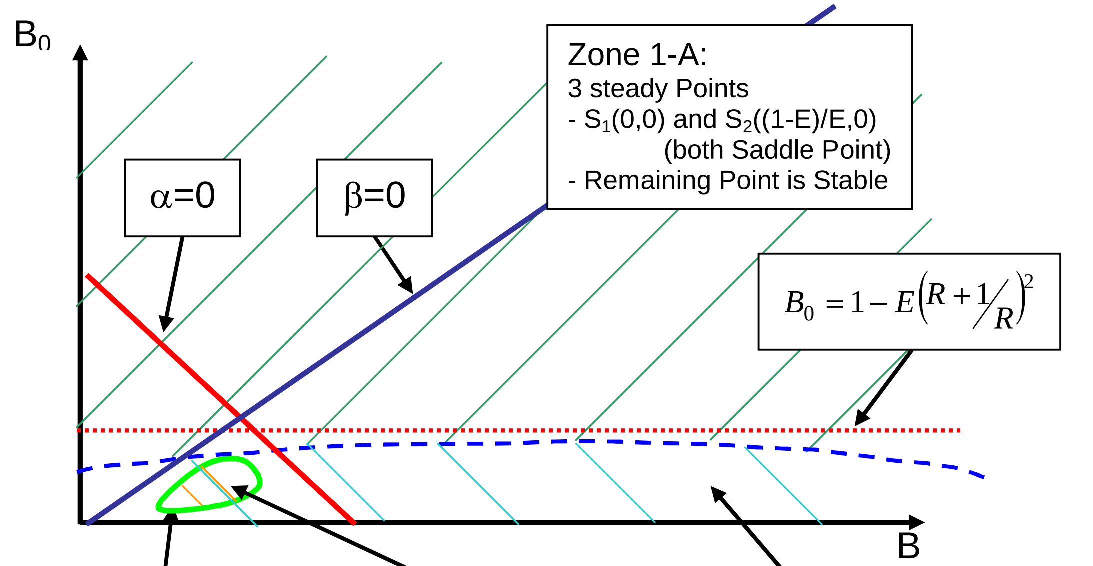


Case 4-1: $E > R^2/(R+1)^2$ and R>1

The condition $E > R^2/(R+1)^2$ ensures that the trace of the Jacobian of the steady points linked to the polynomial P is negative.



Case 4-2: $E \leq R^2/(R+1)^2$ and R>1



Zone 2:
4 steady Points
- $S_1(0,0)$: Saddle Point
- $S_2((1-E)/E,0)$:Idem
- Last Two Points:
  - Unstable if $D<D_{lim}$(point)
  - Center if $D = D_{lim}$(point)
  - Stable if $D>D_{lim}$(point)

NB: For one of the points: stability and Unstability is week (0 Eigenvalue of Jacobian)

Zone 3:
5 steady Points
- $S_1(0,0)$: Saddle Point
- $S_2((1-E)/E,0)$: Idem
- Two Points are:
  - Unstable if $D<D_{lim}$(point)
  - Center if $D = D_{lim}$(point)
  - Stable if $D>D_{lim}$(point)
- Last Point is a Saddle Point

Zone 1-B:
3 steady Points
- $S_1(0,0)$: Saddle Point
- $S_2((1-E)/E,0)$: Idem
- Last Point:
  - Unstable if $D<D_{lim}$
  - Center if $D = D_{lim}$
  - Stable if $D>D_{lim}$

Warning on the Border
I have no clue where it actually is located except it is below the red dotted line. No clue on its shape either…. It is probably made up of several regions to be honest
I have made the diagram easy for myself by assuming that Zone 2 and 3 were entirely below the border, which is false for some combinations of E and R (as E increases the region where the Trace may change size decreases in size and eventually disappears).