Imperial College London

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Department of Bioengineering
MEng/BEng in Biomedical Engineering

BE3-H36/BE4-H36 – Modelling in Biology (MiB)

Assignment 3: *Three- and higher dimensional nonlinear systems: chaos and neurons*

Instructions: To be returned to the General Office by **Friday, 14 December 2007**.
As always, it is OK to discuss with other students and to use the MATLAB Help, but you must write your own answers. These should consist of your calculations plus MATLAB plots.

Happy Holidays and best wishes for 2008!

Note: In this shorter assignment you will explore the dynamic behavior of higher order systems of ODEs through numerical simulation. An important aim is that you do *good, careful numerics* and that you interpret them correctly.

Question 1: A numerical exploration of the Lorenz system

Consider the Lorenz system we saw in class:

$$\begin{aligned} \dot{x} &= \sigma\,(y - x) \\ \dot{y} &= rx - xz - y \\ \dot{z} &= xy - bz, \end{aligned} \tag{1}$$

where we will fix $\sigma = 10$ and $b = 8/3$ throughout.

1. Using `ode45`, investigate numerically the dynamical behavior of the system for five values of r: $r = \{10, 24.5, 25, 45, 220\}$. In each case, plot: (i) $x(t)$ (i.e., the temporal evolution of one of the variables of the system) and (ii) x vs. y (i.e., a 'projection' of the 3D phase space on the plane). Use your numerical results to identify the *long-term attractors* of the system for each value of r?

   [To give a complete answer, it is important that you perform careful numerics, considering many initial conditions, different lengths of integration, and sufficient numerical precision and time sampling.]

2. We will now characterize what we mean with 'sensitivity to initial conditions.' Fix $r = 26$ for (1) and numerically integrate the system with `ode45` from the initial condition $(x_0, y_0, z_0) = (8.8756, 16.1229, 11.5828)$ for `t=[0:0.001:50]`. Plot $x(t)$. On the same figure, plot another trajectory $x(t)$ where everything is the same except the initial condition is now $(x_0, y_0, z_0) = (8.8757, 16.1230, 11.5829)$. At what time can you see the trajectories diverging? Use your numerics to obtain an estimate of the Lyapunov exponent of the system.

Question 2: Period doubling and chaos in an oscillatory system

Some types of neurons (e.g. in the cerebellum) exhibit sub-threshold oscillations. Consider the following very simple system of ODEs, which has been proposed to model such oscillations:

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + \frac{1}{5}y \\ \dot{z} &= \frac{1}{5} + z(x - c). \end{aligned} \tag{2}$$

1. Use `ode45` to investigate numerically the dynamical behavior of the system for $c = \{2.6, 3.5, 4.1, 5\}$. For each value of c, plot a trajectory $x(t)$, and a *phase portrait* of x vs. y. Use your plots to identify the attractors of the system for the different values of c. Explain the observed behavior as c grows, paying special attention to the period of the signal.

2. Refresh your 'Signal processing' knowledge and use Matlab to calculate and plot the power spectrum of the four time signals $x(t)$ that you obtained in 1. Explain briefly the results from the point of view of Fourier analysis.

   [Numerical hint: To improve the plots of the power spectrum, it is very important that you discard the initial transient and that you perform very long time integrations.]

Question 3: The dynamical response of a spiking neuronal model

A recently proposed simplification of the Hodgkin-Huxley equations leads to the following four-dimensional spiking neuronal model:

$$\begin{aligned}
\frac{dV}{dt} &= -m_\infty[V](V-0.5) - 26R(V+0.95) - g_T T(V-1.2) - g_H H(V+0.95) + I \\
\frac{dR}{dt} &= \frac{1}{\tau_R}(-R + R_\infty[V]) \\
\frac{dT}{dt} &= \frac{1}{14}(-T + T_\infty[V]) \\
\frac{dH}{dt} &= \frac{1}{45}(-H + 3T),
\end{aligned} \tag{3}$$

where $m_\infty[V] = 17.8 + 47.6V + 33.8V^2$; $R_\infty[V] = 1.24 + 3.7V + 3.2V^2$; $T_\infty[V] = 8(V + 0.725)^2$ and I is the injected current (in nA). Note that the time t is measured in ms and the voltage is measured in $10^-2\ mV$. The three parameters in the model (τ_R, g_T, g_H) are neuron-dependent and lead to different dynamical responses.

To explore this model, write a Matlab program to simulate this neuron numerically with `ode45` for different sets of parameters and different injected currents. Present your numerics compactly by using the Matlab command `subplot`.

[Numerical hints: The initial condition is quite important. The following initial condition $(V_0, R_0, T_0, H_0) = (-0.8, 0.3, 0.3, 1)$ *seems to give good results. Also, to obtain enough accuracy, you should use* `ode45` *with* `t=[0:1e-3:tfinal]`*]*

1. Consider first a regular spiking (RS) neuron, given by $g_T = 0.1, g_H = 5, \tau_R = 4.2$, subjected to seven different *constant* injected currents $I = \{0.1, 0.2, 0.4, 0.6, 0.8, 1, 1.5\}$. Plot $V(t)$ for `t=[0:1e-3:300]` for the seven currents. Use these time trajectories to estimate the *spiking rate* for each value of I, and plot the spiking rate vs. I. Which of the currents I are sub-threshold?

2. Experimental evidence seems to indicate that inhibitory neurons in the neocortex are exclusively fast spiking (FS) cells. This regime is characterized by $g_T = 0.25, g_H = 0, \tau_R = 1.5$. Repeat the calculations and plots in 1 for a FS neuron with the same injected currents for `t=[0:1e-3:150]`.

3. The main experimental differences between RS and FS neurons are: the spiking rate, the threshold, the resting potential and the shape of the spikes. Use your numerical simulations to compare those characteristics.

4. Another group of neocortical neurons (CB) produce 'continuous bursting' under constant current stimulation. These CB neurons correspond to $g_T = 2.25, g_H = 9.5, \tau_R = 4.2$. Plot $V(t)$ for a CB neuron with a constant injected current $I = 0.8$. Estimate from your plot the inter-spike period and the inter-burst period. Compare it to the spiking rates obtained in 1 and 2 for $I = 0.8$.