Equilibrium analysis for 2-nutrient model

The model:

$$\frac{dy}{dt} = r\frac{Vnc}{(K_n+n)(K_c+c)} - gy \qquad (a)$$

$$\frac{dn}{dt} = gu_n - gn - \frac{Vnc}{(K_n+n)(K_c+c)} \qquad (b)$$

$$\frac{dc}{dt} = gu_c - gc - \frac{Vnc}{(K_n+n)(K_c+c)} \qquad (c)$$

From equation (a) at equilibrium ($\frac{dy}{dt}=0$)
we have $y=0$ OR $\frac{nc}{(K_n+n)(K_c+c)} = g/rV$ (d)

From equation (b) we have $y=0$ and $u_n = n$ OR

$0 = u_n - n - \frac{1}{r}y$ (using (d)) (e)

Likewise from (c) we have $y=0$ and $u_c = c$ OR

$0 = u_c - c - \frac{1}{r}y.$ (f)

One equilibrium is $y=0$, $n=u_n$, $c=u_c$.
Another is defined by (d), (e), (f):

$$y = r(u_n - n) = r(u_c - c)$$

$$\frac{nc}{(K_n+n)(K_c+c)} = \frac{g}{rV}$$

Continuing, we have

$$\frac{nc}{(K_n+n)(K_c+c)} = \frac{g}{rV} \quad \text{and} \quad u_n - n = u_c - c$$

We define $p = \frac{g}{rV}$, noting $p<1$ for (d) to be able to hold.

Next, we let $\Delta u = u_n - u_c$, and we see

$$n = \Delta u + c.$$

Substituting, we have

$$\frac{nc}{(K_n+n)(K_c+c)} = \frac{(\Delta u+c)(c)}{(K_n+\Delta u+c)(K_c+c)} = p$$

$$\Delta u\, c + c^2 = pc^2 + p(K_n+K_c+\Delta u)c + pK_c(K_n+\Delta u)$$

$$(1-p)c^2 + [(1-p)\Delta u - (K_n+K_c)p]c - pK_c(K_n+\Delta u) = 0$$

which we write as

$$Ac^2 + Bc + C = 0,$$

with $A>0$, $C<0$, so that there is one negative solution (which is not physically relevant) and one positive sol'n:

$$c = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

Plugging in for A, B, C', we have

$$c = \frac{-(1-p)\Delta u + K_n + K_c + \sqrt{(-(1-p)\Delta u + K_n + K_c)^2 + 4(1-p)pK_c(K_n + \Delta u)}}{1-p}$$

and $n = \Delta u + c$ and $y = r(u_c - c)$

The steady states are given by these 3 equations,

OR the possible $y = 0,\ n = u_n,\ c = u_c$.

Question 2. How do the steady states depend on the feed rates? We examine this in terms of Δu first.

If $\Delta u = 0$, $\quad c = \dfrac{K_n + K_c + \sqrt{(K_n + K_c)^2 + 4p(1-p)K_cK_n}}{1-p}$

$$n = c\ ,\ y = r(u_c - c)$$

[Note that if this value of $c > u_c$, it's not a valid steady state value & the only physical steady state is $y = 0,\ u_n = n,\ u_c = c$.]

What happens as Δu gets large?

$\lim\limits_{\Delta u \to \infty} c$ looks like $-\infty + \infty$, which is indeterminate. We need to do some work...

$$(K_n+K_c)-(1-\rho)\Delta u + \sqrt{(K_n+K_c-(1-\rho)\Delta u)^2 + 4(1-\rho)\rho K_c(K_n+\Delta u)}$$

$$= \frac{(K_n+K_c-(1-\rho)\Delta u)^2 - (K_n+K_c-(1-\rho)\Delta u)^2 - 4(1-\rho)\rho K_c(K_n+\Delta u)}{K_n+K_c-(1-\rho)\Delta u - \sqrt{[K_n+K_c-(1-\rho)\Delta u]^2 - 4(1-\rho)\rho K_c(K_n+\Delta u)}}$$

$$= \frac{-4(1-\rho)\rho K_c(K_n+\Delta u)}{K_n+K_c-(1-\rho)\Delta u - \sqrt{[(K_n+K_c)-(1-\rho)\Delta u]^2 - 4(1-\rho)\rho K_c(K_n+\Delta u)}}$$

$$= \frac{-\frac{4(1-\rho)\rho K_c K_n}{\Delta u} - 4(1-\rho)\rho K_c}{\frac{K_n+K_c}{\Delta u} - (1-\rho) - \sqrt{\left(\frac{K_n+K_c}{\Delta u^2} - (1-\rho)\right)^2 - \frac{4(1-\rho)\rho K_c(K_n+\Delta u)}{\Delta u^2}}}$$

$$\xrightarrow[\Delta u \to \infty]{} \frac{-4(1-\rho)\rho K_c}{-2(1-\rho)} = 2\rho K_c$$

$\therefore$ $c \to \dfrac{2\rho K_c}{1-\rho}$. If $\Delta u \to \infty$ i $u_n \to \infty$ and u_c = fixed

$$y = r(u_c - c) \to r\left(u_c - \frac{2\rho K_c}{1-\rho}\right).$$

$$n = \Delta u + c \to \infty.$$

Other models?

$$Vry\frac{nc}{K+nc} - qy = 0 \quad \rightarrow \quad y=0 \text{ or } \frac{nc}{K+nc} = \frac{q}{rV}$$

The other equations yield the same:

$$y = r(u_n - n) = r(u_c - c)$$

$$u_n - u_c + c = n$$

we get a very similar, equally painful, result.

How about $\dfrac{n+c}{K+n+c}$?

$$\frac{n+c}{K+n+c} = \frac{q}{rV} \; ; \quad n = c + \Delta u.$$

$$\frac{2c+\Delta u}{2c+K+\Delta u} = \frac{q}{rV} = p$$

$$2c + \Delta u = 2cp + pK + p\Delta u$$

$$2(1-p)c = pK + (p-1)\Delta u$$

$$c = \frac{pK + (p-1)\Delta u}{2(1-p)} \; ; \quad n = \frac{pK + (1-p)\Delta u}{2(1-p)}$$

$y = r(u_c - c)$. Note the linear dependence!

Note c could go negative if Δu large enough — what does that mean??