Homework #1 SOLUTIONS

Question 1

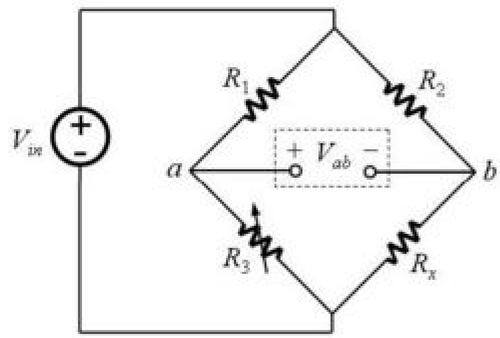

Figure 1a: Wheatstone bridge.

(a) **2 points.** *Assuming R_3 is set such that the bridge is balanced (i.e. V_{ab}=0), derive an analytical expression for R_x in terms of R_1, R_2 and R_3.*

Since $V_{ab} = 0$, V_a and V_b must be equal. Using voltage divider relations gives:

$$V_a = V_{in}\left(\frac{R_3}{R_1+R_3}\right) \quad \text{and} \quad V_b = V_{in}\left(\frac{R_x}{R_2+R_x}\right)$$

Setting these equal to each other and simplifying, we get

$$R_x = \frac{R_2 R_3}{R_1}$$

(b) **3 points.** *Now let R_3 also be a fixed resistor. Suppose that R_x varies in a way that makes V_{ab} nonzero. Derive an expression for the current that would flow if you connected an ammeter from a to b. Assume the ammeter has zero internal resistance.*

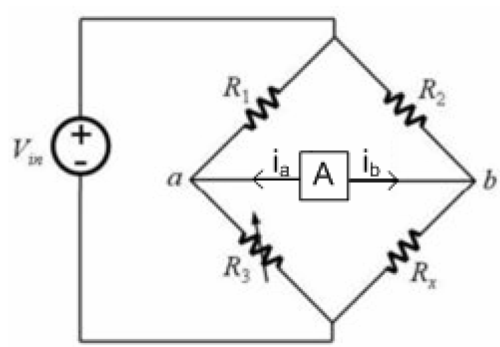

Figure 1b: Wheatstone bridge with ammeter across outputs.

Let the current through the ammeter = $i_a + i_b$.
Once the ammeter connects V_a and V_b, the circuit now looks like:

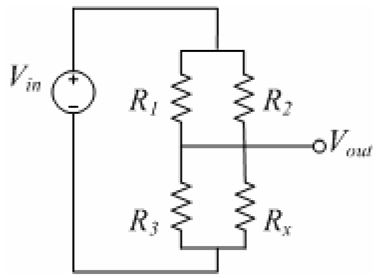

Figure 1c: same circuit as in above parts, with outputs connected by zero resistance ammeter.

V_a and V_b are equal (connected by a zero resistance ammeter), and using the voltage divider relation are equal to:

$$V_a = V_b = V = V_{in}\frac{R_3 \parallel R_x}{R_3 \parallel R_x + R_1 \parallel R_2}$$
$$= V_{in}\frac{R_3 R_x}{R_3 R_x + \frac{(R_3+R_x)R_1 R_2}{(R_1+R_2)}}$$

Using Kirchoff's current law at nodes a and b:

$$\frac{V_{in}-V_a}{R_1} + i_a - \frac{V_a}{R_3} = 0$$
$$\frac{V_{in}-V_b}{R_2} + i_b - \frac{V_b}{R_x} = 0$$

Solving for the current i_a+i_b and plugging in for V gives:

$$i_a + i_b = \frac{-V_{in}}{R_1} + \frac{V}{R_1} + \frac{V}{R_3} - \frac{V_{in}}{R_2} + \frac{V}{R_2} + \frac{V}{R_x}$$

$$= V_{in}\left(\frac{-R_2 - R_1}{R_1 R_2}\right) + V_{in}\left(\frac{R_3 R_x}{R_3 R_x + \frac{(R_3+R_x)R_1 R_2}{(R_1+R_2)}}\right)\left(\frac{R_1 R_3 R_x + R_2 R_3 R_x + R_1 R_2 R_x + R_1 R_3 R_2}{R_1 R_2 R_3 R_x}\right)$$

Which simplifies to:

$$\frac{2V_{in}}{\frac{R_3 R_x}{(R_3+R_x)} + \frac{R_1 R_2}{(R_1+R_2)}}$$

Question 2

…imagine a Wheatstone bridge made out of four identical thermistors, as shown in figure 3. One of the thermistors (R_4) is attached to an odd-looking blue apparatus that varies in temperature. The other three are maintained at a constant 20°C.

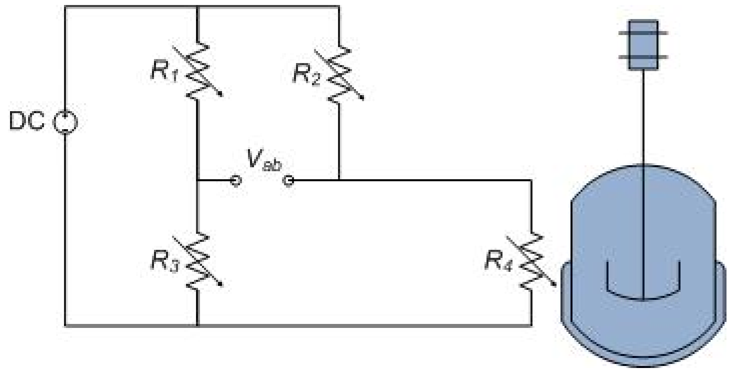

Figure 2a: Wheatstone bridge made of 4 thermistors.

(a) **2 points.** *Derive an expression for V_{ab} as a function of temperature.*

Using the voltages V_a and V_b described earlier,

$$V_a - V_b = \frac{R_3 V_{in}}{R_1 + R_3} - \frac{R_4 V_{in}}{R_2 + R_4}$$

In this part the three resistances R_1, R_2 and R_3 are all constant at:

$$R_1, R_2, R_3 = R_0 + 20\alpha$$

While R_4 varies with temperature:

$$R_4 = R_0 + \alpha T$$

Substituting back into the equation for V_{ab} gives:

$$V_{ab} = \frac{V_{in}(R_0 + 20\alpha)}{2R_0 + 40\alpha} - \frac{V_{in}(R_0 + \alpha T)}{2R_0 + \alpha(20 + T)}$$

The first term here is constant, but the second term varies with temperature.

b) **3 points.** *What if both R_1 and R_4 are attached to the apparatus? Which configuration is more sensitive to temperature variations?*

now $R_1 = R_0 + \alpha T$. Plugging back into the voltage formula gives:

$$V_{ab} = \frac{V_{in}(R_0 + 20\alpha)}{2R_0 + \alpha(20 + T)} - \frac{V_{in}(R_0 + \alpha T)}{2R_0 + \alpha(20 + T)}$$

The second configuration has opposite resistors changing with temperature. So if the resistances both get bigger or smaller, voltages at a and b will move in opposite directions, making the difference, V_{ab} larger.

Question 3

3 points. *Using the data that you collected in the lab for the photodiode, generate 3-4 i-v curves for a photodiode at different light levels (including in darkness). Plot these on the same graph to see how incident light affects diode i-v characteristics.*

2 points. *Give a brief (qualitative) explanation for why photodiodes are best used in reverse bias?*

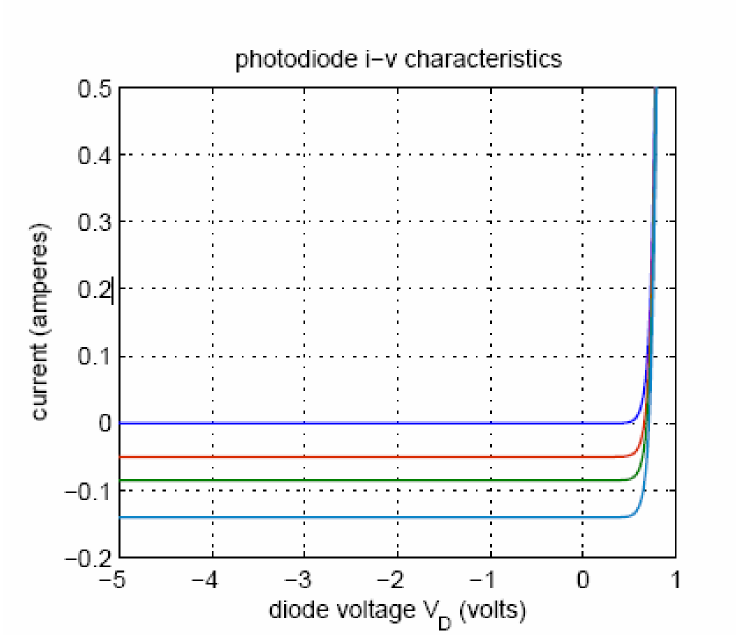

Figure 3: iv-characteristics of a photodiode for varying intensities of light.

Figure 3 shows approximately what you should have found, having taken current and voltage measurements of a photodiode at several different light levels. The topmost curve (with a reverse current of nearly zero) shows the diode with no light reaching it, and the three curves below it show its behavior with increasing light. Its salient behavior is that in reverse bias, current is approximately independent of voltage, and light produces increased reverse current in proportion to incident light power. If you'd like to understand the physical origin of this behavior, take a look at the American Journal of Physics article referenced on the diode tutorial page of the 20.309 site.

Question 4

6 points. The plots and fit functions to all four unknown boxes are shown in Fig. 4. The only unusual one is box "B", having a double-pole low-pass filter (essentially two low-pass filters cascaded). Note its steeper rolloff (40 dB/decade) than box "C", the single-pole low-pass, with the normal 20 dB/decade roll-off.

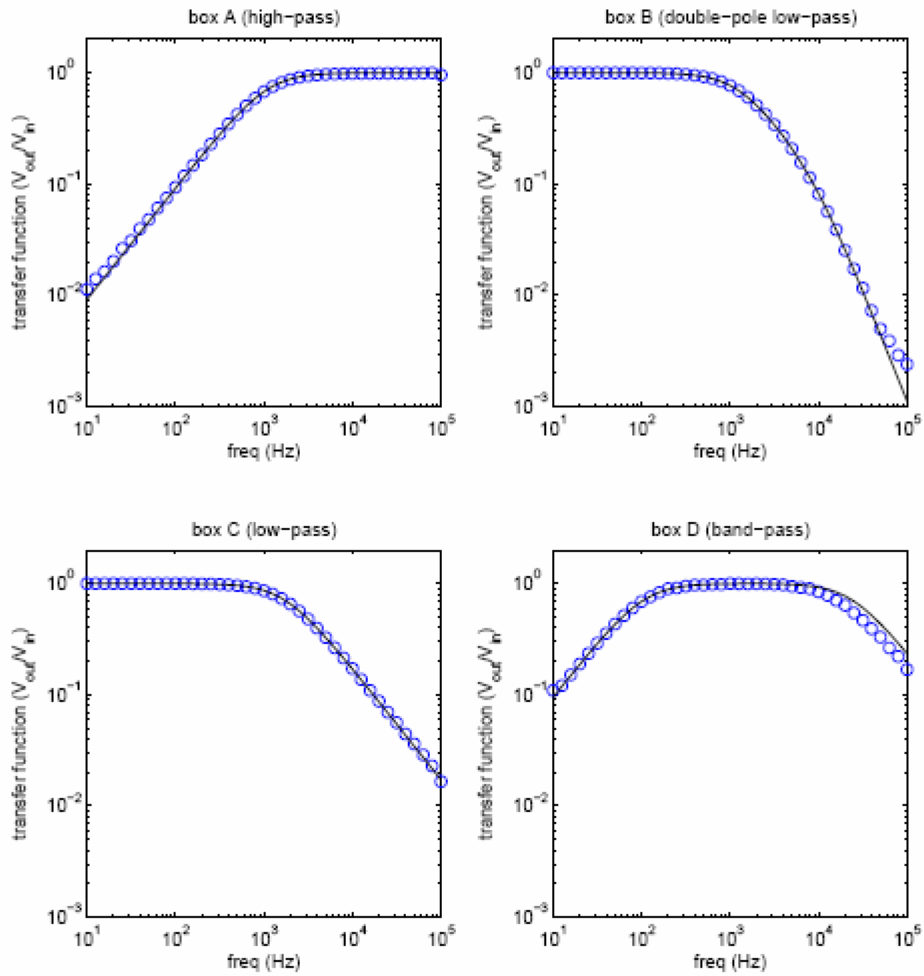

Figure 4: The transfer functions, and corresponding fits of the four "black boxes".

A typical function to fit to this data appears as follows:

```
function output = hpfilter(RCs, x)
cap = RCs(1);
res = RCs(2);
output = 1./((x.*cap.*res).^2+1).^(.5);
```

This is invoked by using

```
Fit = lsqcurvefit(@hpfilter, [10e-6 100], freq, outputNorm);
```

where 10^{-6} and 100 are initial guesses for the capacitor and resistor values, respectively, `freq` is the vector of frequency values, and `outputNorm` is the vector of amplitude values. Note that the fit will not return unique values of *R* and *C* for these circuits, since this problem is actually under-constrained, and many different *R* and *C* combinations can produce the correct corner frequencies for each plot (the correct values can't be determined without additional information).

Question 5

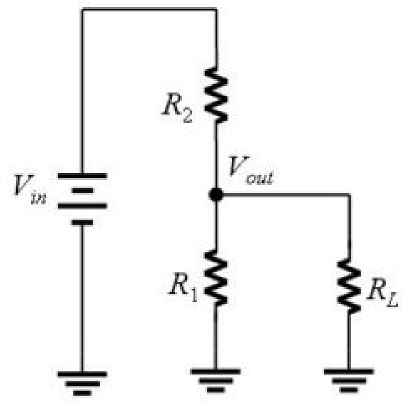

Figure 5a: A typical voltage divider circuit with load, R_L.

5 points *What value of R_L (in terms of R_1 and R_2) will result in the maximum power being dissipated in the load?*

We know from Thevenin's theorem, that we can represent the circuit above as a simpler one:

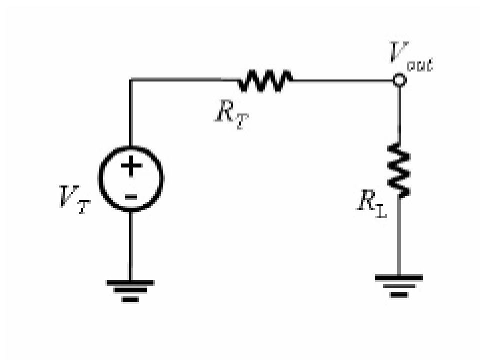

Figure 5b: Thevenin equivalent circuit for the one in Fig. 5a.

In this circuit, the voltage passed to the load is – yet another simple divider relation. Power through the load resistor is given by $P = V_{out}{}^2 / R_L$, so we get

$$P = V_T{}^2 \frac{R_L}{(R_T + R_L)^2}$$

Now it's just a classic local maximum problem from basic calculus – we differentiate (quotient rule!) and set the result equal to zero:

$$\frac{dP}{dR_L} = V_T{}^2 \frac{(R_T + R_L)^2 - 2R_L(R_T + R_L)}{(R_T + R_L)^4} = 0$$

A little simplifying yields $\mathbf{R_L = R_T}$.

The value of R_T is just the parallel combination of R_1 and R_2 from the original circuit, so **maximum power transfer happens when the load resistance is equal to the equivalent output resistance of the source**:

$$R_L = \frac{R_1 R_2}{R_1 + R_2}.$$

(Note that when dealing with instrument signals, we're almost never interested in maximum *power* transfer, but rather maximum *voltage* transfer, which is why it's rarely the case that load and source resistances are matched.)

Question 6

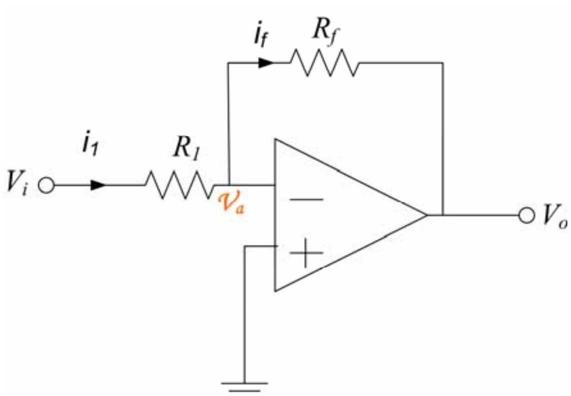

Figure 6a: Inverting Voltage Amplifier

(a) **2 points**. *Calculate the gain of this circuit, V_o/V_i, in terms of the input voltage and the two resistor values.*

By one of the Golden Rules, we know that $i_f = i_1$.
Substiuting voltages and resistances through Ohm's Law gives:

$$\frac{V_a - V_o}{R_f} = \frac{V_i - V_a}{R_1}$$

The other Golden rule says that V_- must equal V_+, so $V_a = 0$:

$$V_a = 0 \Rightarrow \frac{-V_o}{R_f} = \frac{V_i}{R_1}$$

$$\frac{V_o}{V_i} = -\frac{R_f}{R_1}$$

(b) **2 points.** *Derive an expression for the output voltage of the circuit produced by a DC current input at i_{in}. (At DC, you can ignore the affect of the capacitor.) Express your answer in the form of a transfer function, V_{out}/I_{in}.*

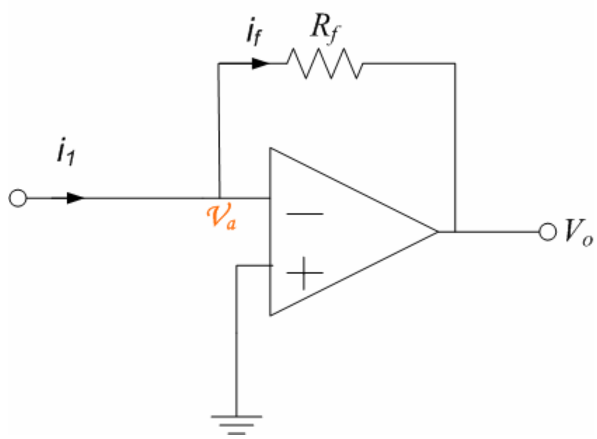

Figure 6b: Transimpedence amplifier at DC (C=0).

We can ignore the capacitor at DC (ω=0), since the impedance ($Z_c = 1/j\omega C$) of it will be extremely large. Using the golden rules and ohm's law again gives,

$$V_a = 0 \Rightarrow i_{in} = i_f$$
$$V_{out} = -i_f R_f$$
$$\frac{V_{out}}{i_f} = -R_f$$

(c) **2 points.** *What is the high frequency gain of the circuit in Figure 6. Remember that a capacitor acts like an open circuit at low frequencies and a short circuit at high frequencies.*

At high frequencies a capacitor is like a shorted wire, so the path of least resistance from V_- to V_{out} has zero resistance, and the gain is 0.

(d) **1 points.** *A transimpedence amplifier with a gain of approximately 10^8 V/A will be required for the DNA lab. What value of resistor in the circuit of Figure 6 would achieve this gain?*

For $|\frac{V_{out}}{i_f}| = 10^8$, need $R_f = 10^8$ (a huge resistance).

(e) **4 points.** *Derive an expression for the output voltage of the circuit in figure 6b in terms of the input current and the three resistor values.*

We can start by labeling the currents through the three resistors i_1, i_2, and i_3, and labeling as x the node to which all three resistors connect, with voltage V_x (see Fig. 6b).

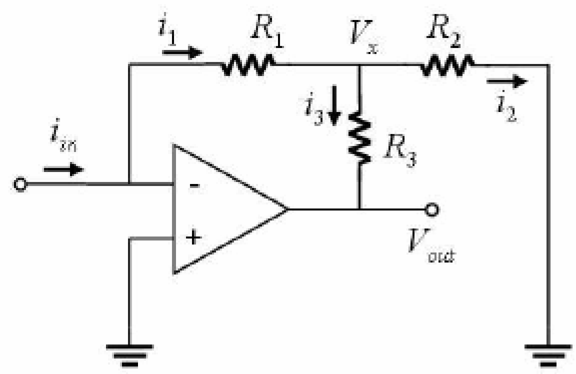

Figure 6b: The circuit for Problem 6(a) with currents labelled.

We now apply the "golden rules" of op-amps, and a few simple circuit laws to write down some expressions:

- Since the voltages at the op-amp's inputs will be kept equal, its (–) input is at ground potential, because the (+) is grounded. Also, because the op-amp inputs draw no current, all of i_{in} passes through R_1, and is therefore equal to i_1.
- By Ohm's Law, $V_x = -i_1 R_1$ (the negative sign reflects the direction of the arrow we drew, relative to the label V_x). Likewise, by Ohm's Law, $V_x = i_2 R_2$ and $V_x - V_{out} = i_3 R_3$.
- For the three currents into/out of node x, Kirchhoff's Current Law (KCL) requires that $i_1 - i_2 - i_3 = 0$ (i_1 is positive, since it's going into the node, as drawn in Fig. 6b, and i_2 and i_3 both flow out of the node, so they are negative), and substituting for the currents, we have

$$i_{in} - \frac{V_x}{R_2} - \frac{(V_x - V_{out})}{R_3} = 0$$

Then, substituting for V_x, we get

$$i_{in} + \frac{i_{in} R_1}{R_2} + \frac{(V_{out} + i_{in} R_1)}{R_3} = 0,$$

and a little more algebra yields

$$\frac{V_{out}}{i_{in}} = -(R_1 + R_3 + \frac{R_1 R_3}{R_2})$$

A few things to note: (1) This is an inverting configuration, meaning that the output signal is the negative of the input. (2) The purpose of this circuit is to enable a very high transimpedance gain without using unreasonably large resistors – to do this, we would choose large values for R_1 and R_3, and a small value for R_2 (e.g. R_1=R_3=100 $k\Omega$, and R_2=100 Ω giving a gain of ≈10^8 V/A). In fact, if the resistors are chosen this way, the first two terms are much smaller than the third and can be neglected, giving simply

$$\frac{V_{out}}{i_{in}} = -\frac{R_1 R_3}{R_2}.$$

(f) **2 points.** *In part C, you determined the effect of putting a capacitor across the feedback resistor in a transimpedence amplifier. High gain amplifiers are succeptible to noise couplig from a variety of sources. Since high frequences are not of interest in the DNA melting lab, it is beneficial to insert a capacitor to reduce the noise.In the circuit of Figure 6b, where would you connect the capacitor and how would you choose its size?*

Our aim here is to achieve a low-pass filter configuration. This turns out to be a non-trivial problem, so we will deal with approximate approaches first:

- One way to think about this is to simply insert the *R*||*C* combination for any of the three resistors in the simplified expression for V_{out}/i_{in} above, and examine the resulting function (as part (c) of this question suggests). The impedance of the *R*||*C* combination is $Z_{RC} = R/(1+j\omega RC)$, which gives a low-pass-type function only when substituted for R_1 or R_3, but not R_2.

  In the case of R_1 we get

$$\frac{V_{out}}{i_{in}} = -\frac{R_1 R_3}{R_2 + j\omega R_1 R_2 C},$$

  whereas in the case of R_2, the result is:

$$\frac{V_{out}}{i_{in}} = \frac{R_1 R_3 + j\omega R_1 R_2 R_3 C}{R_2}.$$

  Thus, one possible capacitor placement is shown in Figure 6c, with the capacitor across R_1.

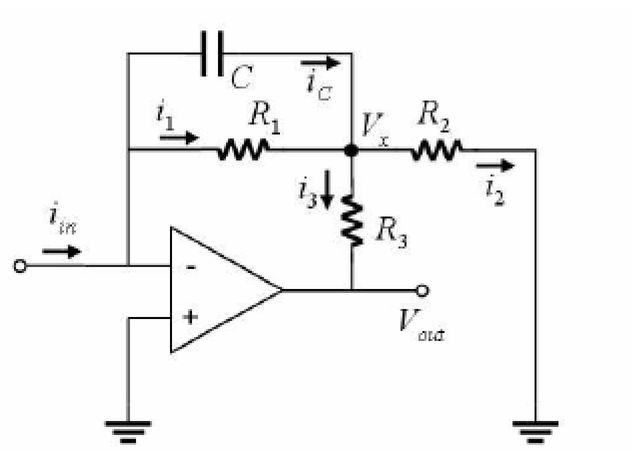

Figure 6c: One possible capacitor placement.

- Another approach is to ask - where can we insert a capacitor in the circuit such that at high-frequency, when it behaves as a short-circuit, it "shorts" the output to ground? The best configuration is shown in Figure 6, with the capacitor across R_1 and R_3. We analyze this in detail below.

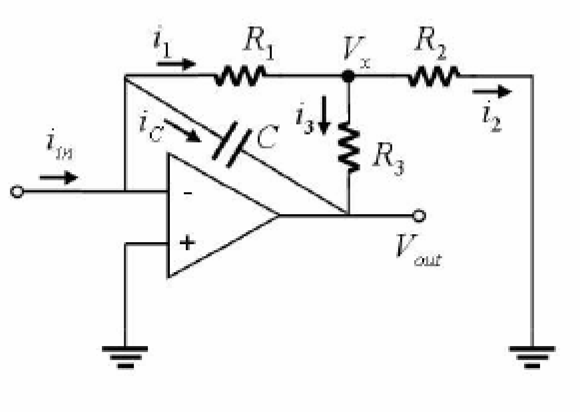

Figure 6d: Best capacitor placement.

(g) **2 points.** *Now write down the expression for this new circuit's output with respect to the current input for AC signals*

- For Figure 6c, KCL at the op-amp (–) input node gives us $i_{in}=i_C+i_1$ and KCL at node x gives $i_c + i_1 = i_2 + i_3$. Using the impedance model, we write $i_C = -V_x/Z_C = -V_x j\omega C$ and we already know that $i_1 = -V_x / R_1$, $i_2 = V_x / R_2$, and $i_3 = (V_x - V_{out})/R_3$.

  Substituting these various currents into the node x KCL equation, we get

  $$-\frac{V_x}{R_1} - V_x j\omega C = \frac{V_x}{R_2} + \frac{V_x - V_{out}}{R_3}$$

  Then, combining the expressions for i_C and i_1 and the (–) node KCL expression, we can write

  $$V_x = -\frac{i_{in} R_1}{1 + j\omega R_1 C}$$

  Finally, inserting V_x into the previous equation, and crunching through the rearrangements, we get

  $$\frac{V_{out}}{i_{in}} = -\frac{R_1 R_3 + R_2 R_3 + R_1 R_2 + j\omega R_1 R_2 R_3 C}{R_2 (1 + j\omega R_1 C)}$$

  This rather unattractive result has one good feature, which is that at DC ($\omega \to 0$) we recover the result from part (a). At high frequency ($\omega \to \infty$), however, the gain simply becomes equal to R_3 – a reduction from $R_1 R_3 / R_2$, but hardly a good low-pass filter.

- For Figure 6d, we again have KCL at the (–) input node of the op-amp giving $i_{in} = i_c + i_1$, and KCL at node x providing $i_1 = i_2 + i_3$. As before, we write the various expressions for branch currents: $i_c = -V_{out} j\omega C$, $i_1 = -V_x / R_1, i_2 = V_x / R_2$, and $i_3 = (V_x - V_{out}) / R_3$.

  Combining i_1 and i_C, and solving for V_x, we get $V_x = -i_{in} R_1 - V_{out} j\omega R_1 C$. Again substituting the currents, as well as the V_x expression into our KCL equation for node x, then doing some rearrangement, we finally obtain:

$$\frac{V_{out}}{i_{in}} = -\frac{R_1 R_3 + R_2 R_3 + R_1 R_2}{R_2 + j\omega C(R_1 R_2 + R_1 R_3 + R_2 R_3)}$$

This expression does have the usual low-pass filter form, and has the behavior we're looking for. At DC, the same expression from part (a) is recovered, and at high ω, the gain goes to 0. The frequencies of interest in the DNA melting lab are a few Hz and below (since the sample cools over several minutes). ω_c occurs where the real and imaginary parts are equal which is at

$$\omega = \sqrt{\frac{R_1 R_2 R_3 + R_2{}^2 R_3 + R_1 R_2{}^2}{C^2 (R_1 R_2 + R_1 R_3 + R_2 R_3}} \ .$$

It is certainly desirable to reduce 60 Hz noise. Therefore, one possible way to choose a capacitor is to make ω_c around 6 Hz. In this case, the 60 Hz noise is attenuated by about 20 dB, which is a factor of 10 (the roll-off of a single pole LPF is 20 dB per decade. If you built a two pole filter, the attenuation at 60 Hz would be 40 dB, or a factor of about 100.)