Homework #1 SOLUTIONS

Problem 1

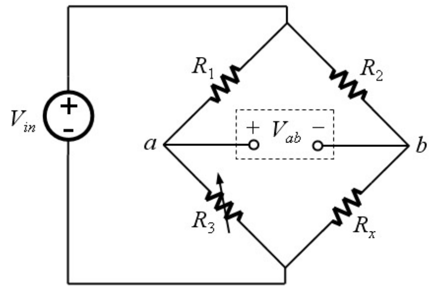


Figure 1: Wheatstone bridge.

(a) *Assuming R_3 is set such that the bridge is balanced (i.e. $V_{ab} = 0$), derive an analytical expression for R_x in terms of R_1, R_2 and R_3.*

Since $V_{ab} = 0$, V_a and V_b must be equal. Using voltage divider relations gives:

$$V_a = V_{in}\left(\frac{R_3}{R_1 + R_3}\right) \qquad \text{and} \qquad V_b = V_{in}\left(\frac{R_x}{R_2 + R_x}\right)$$

Setting these equal to each other and simplifying, we get

$$\frac{R_1}{R_3} = \frac{R_2}{R_x}\,, \qquad \text{or} \qquad R_x = \frac{R_2 R_3}{R_1}\,.$$

(b) *. . . suppose that R_x varies in a way that makes V_{ab} nonzero. Derive an expression for the dependence of V_{ab} on R_x.*

$$V_{ab} = V_b - V_a = V_{in}\left(\frac{R_x}{R_2 + R_x} - \frac{R_3}{R_1 + R_3}\right) \qquad \text{or, if you prefer:} \qquad V_{ab} = V_{in}\left[\frac{R_1 R_x - R_2 R_3}{(R_1 + R_3)(R_2 + R_x)}\right]$$

Problem 2

In what range should the values of R_1, R_2 and R_3 be to make a sensitive measurement? Explain your reasoning.

The aim here is to choose values for the Rs that will give the largest change in V_{ab} for a small change in R_x. Another way to say this is that if we plot the value of V_{ab} as a function of R_x (the output characteristic), we'd like that function to have a slope as steep as possible.

If we only consider the right-hand branch of the Bridge, ignoring the left, we have a voltage divider. Several output curves for it are shown in Figure 2.

It turns out that not only is sensitivity (the line's slope) an issue, but linearity is as well ("flatness" of the output curve). It's clear from the plot that there's a trade-off between sensitivity and linearity. Higher values of R_2 give the most nearly constant slope, while greater sensitivity is available at lower values of R_2, but only for a small range of R_x values.

If we take the left branch of the bridge into account again (with R_1 and R_3) the node V_a simply provides a reference voltage to compare with V_b. Matching the two branches to one another is good practice, but if this isn't possible for some reason (e.g. if higher current is desired in one branch), then R_3 must at least have sufficient range to enable balancing the bridge for typical values of R_x.

Another consideration is the output and input resistance of the Wheatstone branches. E.g. the total equivalent resistance of the Wheatstone network affects how much current drawn, and the power dissipated by it. Also, if the signal voltages V_a and V_b are to be fed into another signal processing stage, their equivalent output impedance ($R_1 \parallel R_3$ and $R_2 \parallel R_x$) shouldn't be too high.

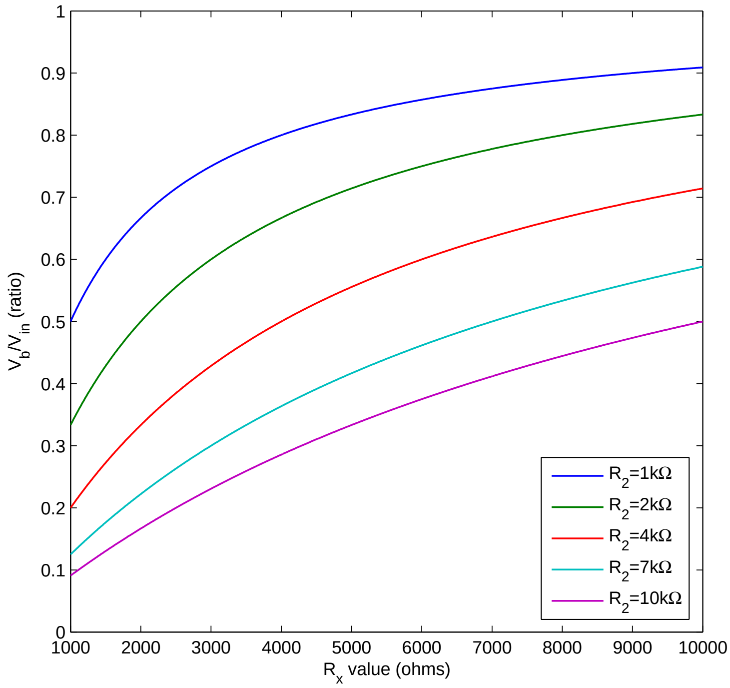


Figure 2: A family of output characteristics, showing the dependence of V_b on R_x at several values of R_2.

In sum, to select good resistor values, it's critical to know the full expected travel range of R_x, as well as the full context in which the circuit will be used.

Problem 3

Figure 3 shows approximately what you should have found, having taken current and voltage measurements of a photodiode at several different light levels. The topmost curve (with a reverse current of nearly zero) shows the diode with no light reaching it, and the three curves below it show its behavior with increasing light. Its salient behavior is that in reverse bias, current is approximately independent of voltage, and light produces increased reverse current in proportion to incident light power. If you'd like to understand the physical origin of this behavior, take a look at the American Journal of Physics article referenced on the diode tutorial page of the 20.309 site, or ask your lab instructor.

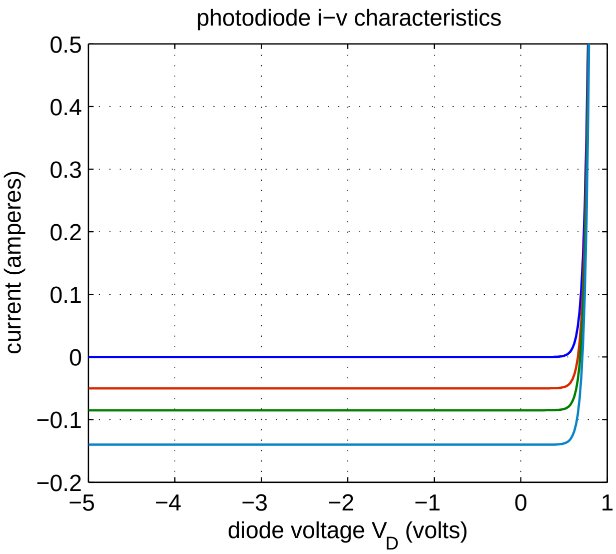


Figure 3: Diode $i - v$ curves (idealized sketches, not actual data).

Problem 4

The plots and fit functions to all four unknown boxes are shown in Fig. 4. The only unusual one is box "B", having a double-pole low-pass filter (essentially two low-pass filters cascaded). Note its steeper rolloff (40 dB/decade) than box "C", the single-pole low-pass, with the normal 20 dB/decade roll-off.

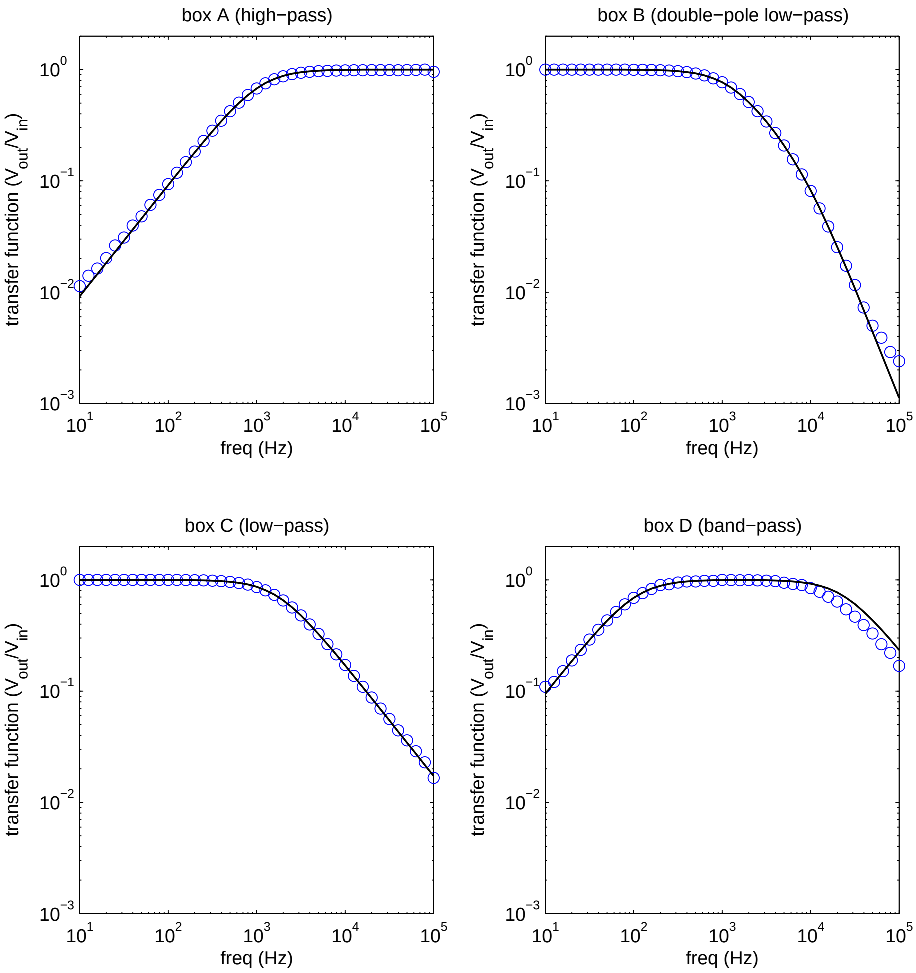


Figure 4: The transfer functions, and corresponding fits of the four "black boxes".

A typical function to fit to this data appears as follows:

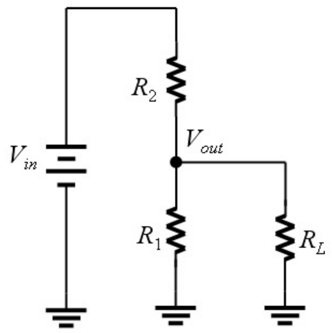
```
function output = hpfilter(RCs, x)
cap = RCs(1);
res = RCs(2);
output = 1./((x.*cap.*res).^2+1).^(.5);
```

This is invoked by using

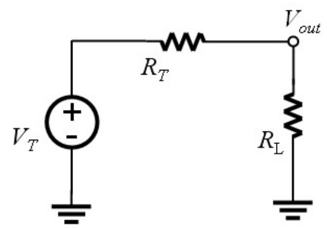
```
Fit = lsqcurvefit(@hpfilter, [10e-6 100], freq, outputNorm);
```

where 10^{-6} and 100 are initial guesses for the capacitor and resistor values, respectively, `freq` is the vector of frequency values, and `outputNorm` is the vector of amplitude values. Note that the fit will not return unique values of R and C for these circuits, since this problem is actually under-constrained, and many different R and C combinations can produce the correct corner frequencies for each plot (the correct values can't be determined without additional information).

Problem 5



What value of R_L (in terms of R_1 and R_2) will result in the maximum power being dissipated in the load?
We know from Thevenin's theorem, that we can represent the circuit above as a simpler one:



In this circuit, the voltage passed to the load is $V_{out} = V_T R_L/(R_T + R_L)$ – yet another simple divider relation. Power through the load resistor is given by $p = V_{out}^2/R_L$, so we get

$$p = V_T^2 \cdot \frac{R_L}{(R_T + R_L)^2} .$$

Now it's just a classic local maximum problem from basic calculus – we differentiate (quotient rule!) and set the result equal to zero:

$$\frac{dp}{dR_L} = V_T^2 \cdot \frac{(R_T + R_L)^2 - 2R_L(R_T + R_L)}{(R_T + R_L)^4} = 0 .$$

A little simplifying yields $R_L = R_T$.

The value of R_T is just the parallel combination of R_1 and R_2 from the original circuit, so **maximum power transfer happens when the load resistance is equal to the equivalent output resistance of the source**:

$$R_L = \frac{R_1 R_2}{R_1 + R_2} .$$

(Note that when dealing with instrument signals, we're almost never interested in maximum *power* transfer, but rather maximum *voltage* transfer, which is why it's rarely the case that load and source resistances are matched.)

Problem 6

(a) *Determine an expression for the output voltage of the circuit with respect to a DC current input. Express your answer in V_{out}/i_{in}.*

We can start by labeling the currents through the three resistors i_1, i_2, and i_3, and labeling as x the node to which all three resistors connect, with voltage V_x (see Fig. 5).

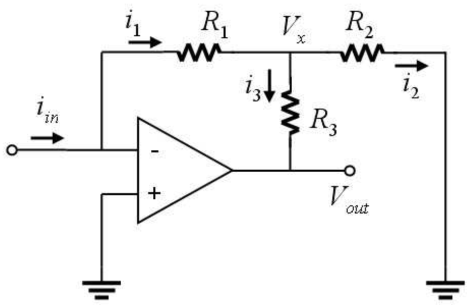


Figure 5: The circuit for Problem 6(a) with currents labelled.

We now apply the "golden rules" of op-amps, and a few simple circuit laws to write down some expressions:

- Since the voltages at the op-amp's inputs will be kept equal, its (–) input is at ground potential, because the (+) is grounded. Also, because the op-amp inputs draw no current, all of i_{in} passes through R_1, and is therefore equal to i_1.
- By Ohm's Law, $V_x = -i_1 R_1$ (the negative sign reflects the direction of the arrow we drew, relative to the label V_x). Likewise, by Ohm's Law, $V_x = i_2 R_2$ and $V_x - V_{out} = i_3 R_3$.
- For the three currents into/out of node x, Kirchhoff's Current Law (KCL) requires that $i_1 - i_2 - i_3 = 0$ (i_1 is positive, since it's going into the node, as drawn in Fig. 5, and i_2 and i_3 both flow out of the node, so they are negative), and substituting for the currents, we have

$$i_{in} - \frac{V_x}{R_2} - \frac{(V_x - V_{out})}{R_3} = 0 \; .$$

Then, substituting for V_x, we get

$$i_{in} + \frac{i_{in} R_1}{R_2} + \frac{(V_{out} + i_{in} R_1)}{R_3} = 0 \; ,$$

and a little more algebra yields

$$\frac{V_{out}}{i_{in}} = -\left(R_1 + R_3 + \frac{R_1 R_3}{R_2} \right) \; .$$

A few things to note: (1) This is an inverting configuration, meaning that the output signal is the negative of the input. (2) The purpose of this circuit is to enable a very high transimpedance gain without using unreasonably large resistors – to do this, we would choose large values for R_1 and R_3, and a small value for R_2 (e.g. $R_1 = R_3 = 100\ k\Omega$, and $R_2 = 100\ \Omega$ giving a gain of $\approx 10^8$ V/A). In fact, if the resistors are chosen this way, the first two terms are much smaller than the third and can be neglected, giving simply

$$\frac{V_{out}}{i_{in}} = -\frac{R_1 R_3}{R_2} \; .$$

(b) *Since this is such a high-gain circuit, it can be quite noisy, if the input current i_{in} experiences high-frequency fluctuations. You can insert a capacitor across one of the resistors to reduce the noise (i.e. make a low-pass filter to eliminate high-frequency content). Where would you insert it, and how would you choose its size?*

Our aim here is to achieve a low-pass filter configuration. This turns out to be a non-trivial problem, so we will deal with approximate approaches first:

- One way to think about this is to simply insert the $R \parallel C$ combination for any of the three resistors in the simplified expression for V_{out}/i_{in} above, and examine the resulting function (as part (c) of this question suggests). The impedance of the $R \parallel C$ combination is $Z_{RC} = R/(1 + j\omega RC)$, which gives a low-pass-type function only when substituted for R_1 or R_3, but not R_2.
  In the case of R_1 we get

$$\frac{V_{out}}{i_{in}} = -\frac{R_1 R_3}{R_2 + j\omega R_1 R_2 C} \ ,$$

  whereas in the case of R_2, the result is

$$\frac{V_{out}}{i_{in}} = -\frac{R_1 R_3 + j\omega R_1 R_2 R_3 C}{R_2} \ .$$

  Thus, one possible capacitor placement is shown in Figure 6, with the capacitor across R_1.

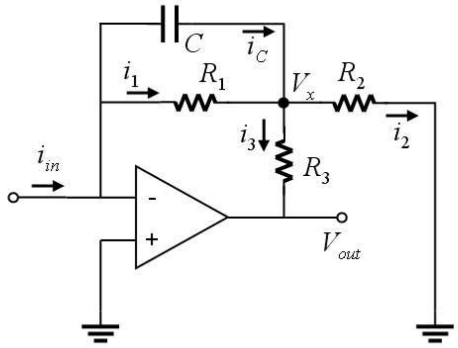


Figure 6: One possible capacitor placement.

- Another approach is to ask - where can we insert a capacitor in the circuit such that at high-frequency, when it behaves as a short-circuit, it "shorts" the output to ground? The best configuration is shown in Figure 7, with the capacitor across R_1 and R_3. We analyze this in detail below.

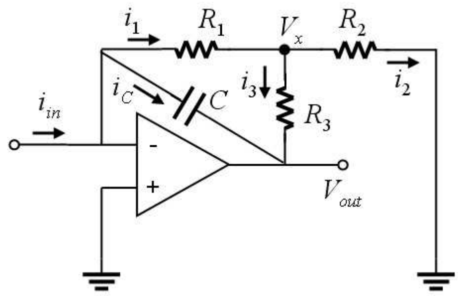


Figure 7: Best capacitor placement.

(c) *Now write down the expression for this new circuit's output with respect to the current input for AC signals (Hint: in the expression from part (a), substitute the parallel combination $R_x \parallel C$ for the resistor R_x that you chose).*

We have already seen the simple case, if we do not consider the full V_{out}/i_{in} expression. You were not expected to do more than this for your homework. For completeness, let's examine the two cases shown in Figures 6 and 7 above.

- For Figure 6, KCL at the op-amp (–) input node gives us $i_{in} = i_C + i_1$ and KCL at node x gives $i_C + i_1 = i_2 + i_3$. Using the impedance model, we write $i_C = -V_x/Z_C = -V_x j\omega C$ and we already know that $i_1 = -V_x/R_1$, $i_2 = V_x/R_2$, and $i_3 = (V_x - V_{out})/R_3$.

  Substituting these various currents into the node x KCL equation, we get

  $$-\frac{V_x}{R_1} - V_x j\omega C = \frac{V_x}{R_2} + \frac{V_x - V_{out}}{R_3}$$

  Then, combining the expressions for i_C and i_1 and the (–) node KCL expression, we can write

  $$V_x = -\frac{i_{in}R_1}{1 + j\omega R_1 C} \ .$$

  Finally, inserting V_x into the previous equation, and crunching through the rearrangements, we get

  $$\frac{V_{out}}{i_{in}} = -\frac{R_1R_3 + R_2R_3 + R_1R_2 + j\omega R_1R_2R_3C}{R_2(1 + j\omega R_1C)}$$

  This rather unattractive result has one good feature, which is that at DC ($\omega \to 0$) we recover the result from part (a). At high frequency ($\omega \to \infty$), however, the gain simply becomes equal to R_3 – a reduction from R_1R_3/R_2, but hardly a good low-pass filter.

- For Figure 7, we again have KCL at the (–) input node of the op-amp giving $i_{in} = i_C + i_1$, and KCL at node x providing $i_1 = i_2 + i_3$. As before, we write the various expressions for branch currents: $i_C = -V_{out} j\omega C$, $i_1 = -V_x/R_1$, $i_2 = V_x/R_2$, and $i_3 = (V_x - V_{out})/R_3$.

  Combining i_1 and i_C, and solving for V_x, we get $V_x = -i_{in}R_1 - V_{out} j\omega R_1 C$. Again substituting the currents, as well as the V_x expression into our KCL equation for node x, then doing some rearrangement, we finally obtain:

  $$\frac{V_{out}}{i_{in}} = -\frac{R_1R_3 + R_2R_3 + R_1R_2}{R_2 + j\omega C(R_1R_2 + R_1R_3 + R_2R_3)}$$

  This expression does have the usual low-pass filter form, and has the behavior we're looking for. At DC, the same expression from part (a) is recovered, and at high ω, the gain goes to 0.

  Choosing the appropriate corner frequency for this circuit is quite tricky analytically, so we will limit ourselves to saying that we'd like to cut out noise at 60 Hz, so choosing a sufficiently large capacitor to place the corner frequency below this should not be difficult (especially given the large resistors used).