Biol 398/Math 388 Week 5 Assignment:
Multiple Nutrients in the Chemostat

Background. Recall our first model of the population dynamics of yeast and a single nutrient, which led to the equations

$$\frac{dc}{dt} = qu - qc - yV_{\max}\frac{c}{K+c}$$
$$\frac{dy}{dt} = yrV_{\max}\frac{c}{K+c} - qy$$

As we discussed in class, this model includes

- State (or dependent) variables c and y, denoting the concentration of nutrient and yeast, respectively;
- Independent variable t denoting time; and
- Parameters q,u,V_{max},K, and r.

In the "c equation," we have two terms that represent the rate at which nutrient is fed into the chemostat and the rate at which residual nutrient is drained out. The third term, closely connected to the first term of the "y equation," denotes the rate at which the nutrient is taken up by the yeast and converted into "new yeast." This rate model is approximately linear for low levels of nutrient concentration but reaches a saturation level, at which additional nutrient no longer increases the reproduction rate of the yeast. The first term in the "y equation" differs from the corresponding third term of the "c equation" only with the parameter r, which models the relative efficiency of turning nutrient mass into yeast mass.

It is important to distinguish between parameters that may be set by the experimenter and those that are properties of the underlying experimental system and not under our direct control.

The parameter q denotes the dilution rate (1/time). If the volumetric inflow rate is Q (vol/time), then the dilution rate is $q= Q/V$ in units of (1/time), where the volume of the mixture in the tank is V. The parameter q is thus controlled directly by the experimentalist.

The parameter u denotes the concentration of the nutrient fed into the reactor, and it is also controlled directly by the experimentalist.

In class, we looked at the equilibrium values of this dynamical system. One obvious equilibrium is $y = 0, c = u$, which represents a situation with no yeast and with nutrient concentration saturating at its in-flow concentration. A second, more interesting equilibrium may occur when the concentrations are at

$$c^* = \frac{qK}{rV_{\max} - q}$$
$$y^* = r(u - c^*)$$

equations that require certain constraints on the parameters (because, e.g., concentrations must be non-negative).

Note that the residual nutrient concentration does NOT depend on the feed concentration of the nutrient!!!! In this model, increasing the feed concentration results in a corresponding linear increase in the steady-state yeast concentration. Since this result is not what we saw in the journal club paper, we are seeking a more refined model.

How do we add a second nutrient? We started this process in the previous assignment, and now we continue. We will look at three models of interest.

Model One: Additive nutrient use.

$$\frac{dc_1}{dt} = qu_1 - qc_1 - yV_1 \frac{c_1 + c_2}{K_1 + c_1 + c_2}$$
$$\frac{dc_2}{dt} = qu_2 - qc_2 - yV_2 \frac{c_1 + c_2}{K_1 + c_1 + c_2}$$
$$\frac{dy}{dt} = yR \frac{c_1 + c_2}{K_1 + c_1 + c_2} - qy$$

The two nutrients are denoted c_1, c_2, respectively. There are some new parameters, and the indexing of parameters has been added to connect them to the nutrients. The parameter *R* replaces the relative efficiency because two nutrients are now required.

The additive model is a type of "or" model: the yeast in this model require one or the other of the nutrients, and if one of them goes to zero, the yeast reproduction is maintained in the presence of the other nutrient. Both nutrients must be depleted before the yeast reproduction term goes to zero. Why doesn't this model represent the journal club paper's situation?

Model Two: Multiplicative nutrient use.

$$\frac{dc_1}{dt} = qu_1 - qc_1 - yV_1 \frac{c_1}{K_1 + c_1} \frac{c_2}{K_2 + c_2}$$
$$\frac{dc_2}{dt} = qu_2 - qc_2 - yV_2 \frac{c_1}{K_1 + c_1} \frac{c_2}{K_2 + c_2}$$
$$\frac{dy}{dt} = yR \frac{c_1}{K_1 + c_1} \frac{c_2}{K_2 + c_2} - qy$$

The two nutrients are denoted c_1, c_2, respectively. There are some new parameters, and the indexing of parameters has been added to connect them to the nutrients. The parameter *R* replaces the relative efficiency because two nutrients are now required.

The product model is a type of "and" model: the yeast in this model require both nutrients, and if one of them goes to zero, the yeast reproduction rate goes to 0 too.

Model Three: Rate limited nutrient use.

$$\frac{dc_1}{dt} = qu_1 - qc_1 - yV_1 \mathbf{min}\left\{\frac{c_1}{K_1 + c_1}, \frac{c_2}{K_2 + c_2}\right\}$$

$$\frac{dc_2}{dt} = qu_2 - qc_2 - yV_2 \mathbf{min}\left\{\frac{c_1}{K_1 + c_1}, \frac{c_2}{K_2 + c_2}\right\}$$

$$\frac{dy}{dt} = yR\,\mathbf{min}\left\{\frac{c_1}{K_1 + c_1}, \frac{c_2}{K_2 + c_2}\right\} - qy$$

The two nutrients are denoted c_1, c_2, respectively. There are some new parameters, and the indexing of parameters has been added to connect them to the nutrients. The parameter *R* replaces the relative efficiency because two nutrients are now required.

The product model is another type of "and" model: the yeast in this model require both nutrients. The yeast growth rate is limited to minimum of the individual conversion rate functions.

The assignment.

(1) Explain why Model One, the additive model, may not be the best choice.

(2) Work out the algebra to show that Model Two's "non trivial" equilibrium is characterized by the equations below, for yeast, residual ammonium, and residual glucose, respectively.

$$y^* = \frac{R}{V_1}\left(u_1 - c_1^*\right)$$

$$y^* = \frac{R}{V_2}\left(u_2 - c_2^*\right)$$

$$\frac{q}{R} = \frac{c_1^*}{K_1 + c_1^*}\frac{c_2^*}{K_2 + c_2^*}$$

(3) Set $q = 0.15, V_1 = 110, V_2 = 180, K_1 = 2.5, K_2 = 10, R = 15, u_2 = 100$. With these values, and the values of u_1 in the journal club paper for ammonium feed concentrations, find the equilibrium values of yeast, residual ammonium, and residual glucose. Recreate the journal club paper Figure 1(a) plot with your findings.

(4) Are there analogous equations to the ones of Part (2) for Model Three? If so, what are they?