Part 5: Stability Analysis of the System

We use the dimension-less version of the system.

$$\begin{cases} \dfrac{dU}{ds} = \dfrac{U}{1+U} - \dfrac{BUV}{B_0 + U} \\ \dfrac{dV}{ds} = \dfrac{CUV}{1+UV} - DV \end{cases}$$

The Jacobian of the system is

$$\boxed{J(U,V) = \begin{pmatrix} \dfrac{1}{(1+U)^2} - \dfrac{BB_0V}{(B_0+U)^2} & -\dfrac{BU}{B_0+U} \\ \dfrac{CV}{(1+UV)^2} & \dfrac{CU}{(1+UV)^2} - D \end{pmatrix}}$$

1) First Steady Point S_1

The Jacobian at the point S_1 is

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -D \end{pmatrix}$$

Therefore S_1 is an unstable saddle point

NB: The larger the dissipative parameter D is, the more stable along V the system is – as we expected.

2) Other Steady Point(s)

We are going to modify the expression of the Jacobian of the system

$$J(U,V) = \begin{pmatrix} \dfrac{1}{(1+U)^2} - \dfrac{BB_0V}{(B_0+U)^2} & -\dfrac{BU}{B_0+U} \\ \dfrac{CV}{(1+UV)^2} & \dfrac{CU}{(1+UV)^2} - D \end{pmatrix}$$

The coordinates of the other Steady Points are linked by the system of equations

$$(\otimes) \Leftrightarrow \begin{cases} BV = \dfrac{B_0 + U}{1+U} \\ V = R - \dfrac{1}{U} \end{cases}$$

It follows that $\dfrac{1}{1+UV} = \dfrac{D}{CU}$ and $\begin{cases} \dfrac{CU}{(1+UV)^2} - D = \dfrac{D^2}{CU} - D = \dfrac{D^2}{C}\left(\dfrac{1}{U} - R\right) = -\dfrac{D^2}{C}V \\ \dfrac{CV}{(1+UV)^2} = \dfrac{D^2 V}{CU^2} \end{cases}$

$$J_{11} = \frac{1}{(1+U)^2} - \frac{BB_0 V}{(B_0+U)^2} = \frac{1}{(1+U)^2} - \frac{B_0}{(B_0+U)^2}\left(\frac{B_0+U}{1+U}\right)$$

that is

$$J_{11} = \frac{1}{(1+U)^2} - \frac{B_0}{(B_0+U)}\left(\frac{1}{1+U}\right) = \frac{1}{(1+U)^2}\left(1 - \frac{B_0(1+U)}{(B_0+U)}\right)$$

and finally

$$J_{11} = \frac{U}{(1+U)^2}\left(\frac{1-B_0}{(B_0+U)}\right)$$

Eventually we get the friendlier form of the Jacobian

$$J(U,V) = \begin{pmatrix} \dfrac{U}{(B_0+U)}\left(\dfrac{1-B_0}{(1+U)^2}\right) & -\dfrac{BU}{B_0+U} \\ \dfrac{D^2 V}{CU^2} & -\dfrac{D^2 V}{C} \end{pmatrix}$$

The Jacobian is very complex. We therefore split the stability study in two parts:

- Study of the Sign of the Determinant
- Study of the sign of the Trace

3) Study of the Determinant

$$Det(J) = \frac{D^2UV}{C(B_0+U)}\left(\frac{B}{U^2} - \left(\frac{1-B_0}{(1+U)^2}\right)\right)$$

Det (J) has the same sign as $\frac{B}{U^2} - \left(\frac{1-B_0}{(1+U)^2}\right)$, hence same sign as $B(1+U)^2 - (1-B_0)U^2$.

Det (J) has same sign as $Q(U) = U^2(B+B_0-1)+2BU+B$

The study of the sign of the determinant can be found in the scans available on this website. The main results are:

Case 1: R<1

There is another steady point (other than S1).
For this steady point , Det (J) is strictly positive

Case 2: R>1 B>1/R

There is another steady point (other than S1).
For this steady point , Det (J) is strictly positive

Case 3: R>1 B=1/R and B_0<(R-1)/R

There is another steady point (other than S1).
For this steady point , Det (J) is strictly positive

Case 4: R>1 B<1/R and $B_0 \leq B(R-1) - 2\sqrt{B(1-BR)}$

There are two steady point (other than S1).
For one this steady point , Det (J) is strictly positive.
For the other steady point Det (J) is strictly negative

4) Study of the Trace

The trace of the Jacobian is $Tr(J) = \frac{U}{(B_0+U)}\left(\frac{1-B_0}{(1+U)^2}\right) - \frac{DV}{R}$

which can be modified as $Tr(J) = \frac{U}{(B_0+U)}\left(\frac{1-B_0}{(1+U)^2}\right) - \frac{D}{BR}\left(\frac{B_0+U}{1+U}\right)$

The trace therefore has the same sign as $\frac{UBR}{(B_0+U)^2}\left(\frac{1-B_0}{(1+U)}\right) - D$

We obtain a simple result on the sign of the trace of the Jacobian :

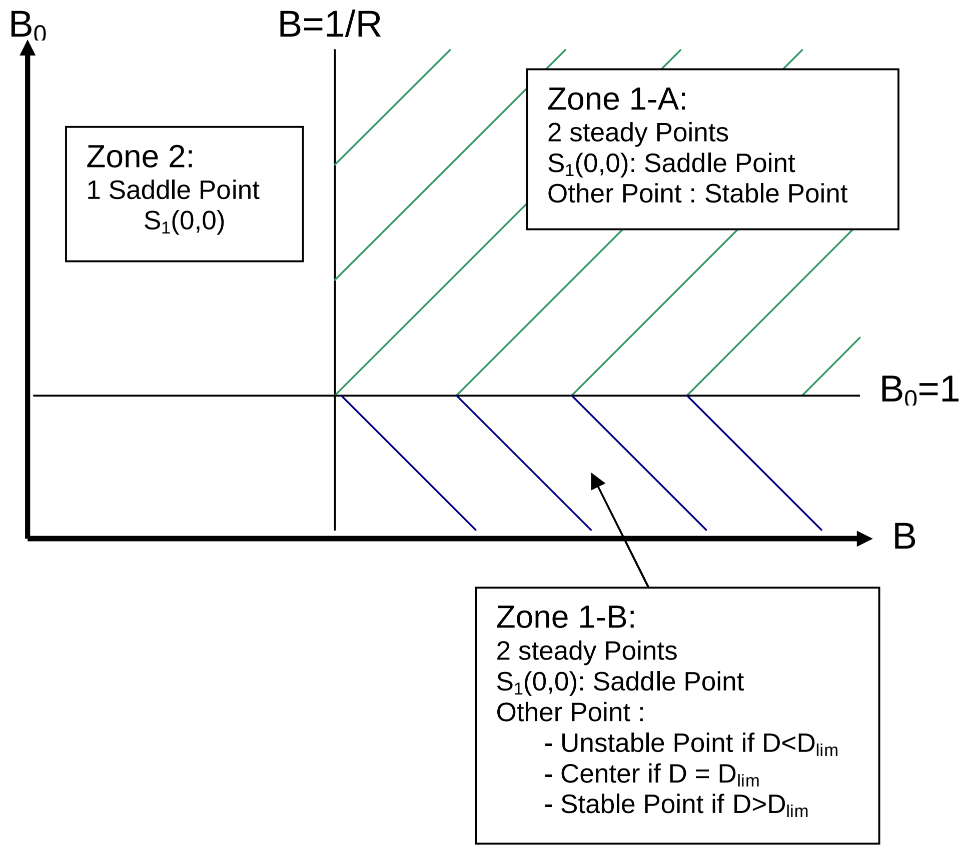
If $B_0>1$ then Tr(J) is strictly negative regardless of D
Conversely if $B_0 \leq 1$ the sign of the Trace changes as D increases

- If $D < D_{\lim}$ then Tr(J) is strictly positive
- If $D = D_{\lim}$ then Tr(J) =0
- If $D > D_{\lim}$ then Tr(J) <0

where the limit value of D is $D_{\lim} = \frac{UBR}{(B_0+U)^2}\left(\frac{1-B_0}{(1+U)}\right)$

5) Graphic Summary of the Stability Analysis

First Case $R \leq 1$



Second Case R>1