Chapter 5

We will examine 2D linear systems to construct our framework for classifying fixed points in 2D nonlinear systems.

remember our general framework of a system of ODE'S ....

$$\frac{dx_1}{dt} = f(x_1, x_2 \dots x_n)$$

$$\frac{dx_2}{dt} = f(x_1, x_2 \dots x_n)$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{dx_n}{dt} = f(x_1, x_2 \dots x_n)$$

$n \rightarrow$ the dimension of our system

so a 2D linear system looks like

$$\frac{dx_1}{dt} = a x_1 + b x_2$$

$$\frac{dx_2}{dt} = c x_1 + d x_2$$

or in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

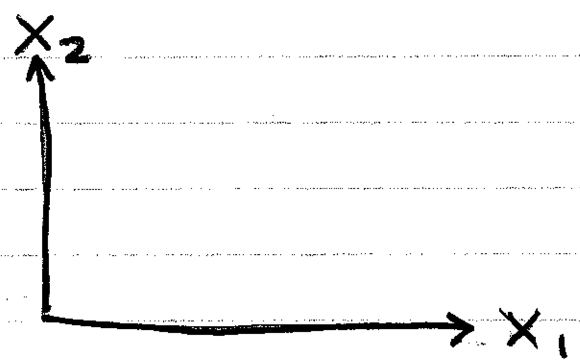
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ is always a fixed point because $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ when $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Phase Plane

for a 1D system we looked at flow on a line
For a 2D system we look at flow on a plane

we have 2 trajectories now: $x_1(t)$ + $x_2(t)$

1D phase space
or state space

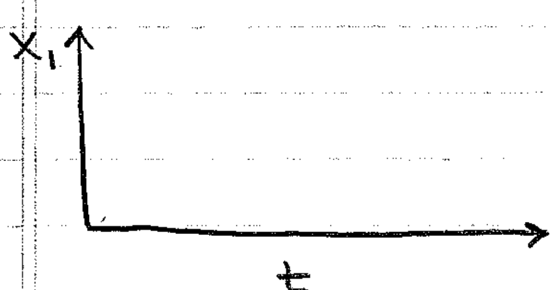


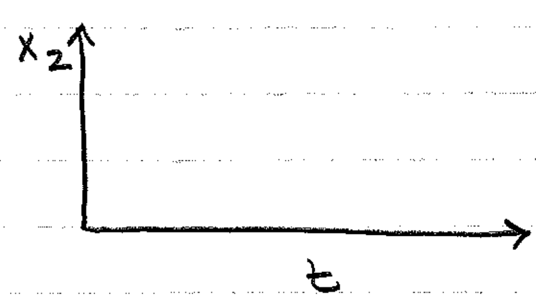
2D phase space
or state space

remember these views of the trajectories, $x_1(t)$ + $x_2(t)$, are called the
phase portraits.

---

as opposed to traditional views of the trajectories.





Vector Fields in 2D $(\dot{x}_1, \dot{x}_2)$

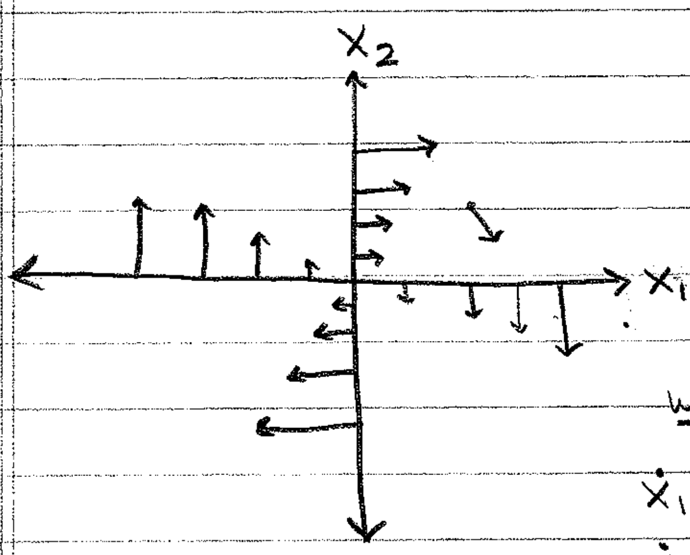
recall that the state of the system (x_1, x_2) at a given t determines how the state variables $x_1 + x_2$ will change in the near future because $(\dot{x}_1, \dot{x}_2)$ is determined uniquely by (x_1, x_2)

so each point on the phase space has a unique velocity vector $(\dot{x}_1, \dot{x}_2)$

take the harmonic oscillator example 5.1.1

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{k}{m} x_1 \qquad \begin{array}{l} k > 0 \ (\text{spring constant}) \\ m > 0 \ (\text{mass}) \end{array}$$

can construct the vector fields simply by plugging in values of the phase space.



<u>when $x_2 = 0$</u> $\quad \dot{x}_1 = 0$

$\dot{x}_2 = -\frac{k}{m} x_1$

$x_1 > 0 \ \downarrow \quad x_1 < 0 \ \uparrow$

<u>when $x_1 = 0$</u> $\quad \dot{x}_1 = x_2$

$\dot{x}_2 = 0$

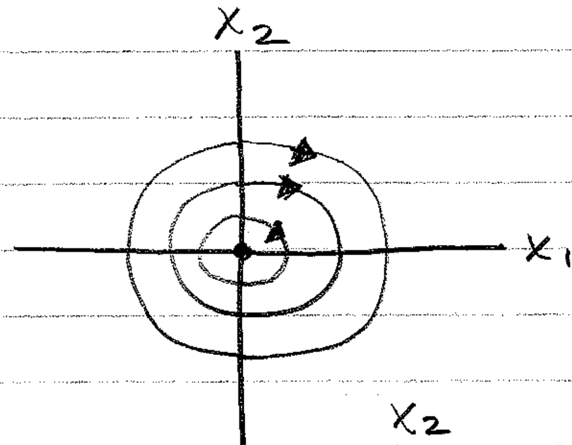
<u>when $x_1 > 0 \quad x_2 > 0$</u>

$\dot{x}_1 = x_2 > 0 \quad \longrightarrow$

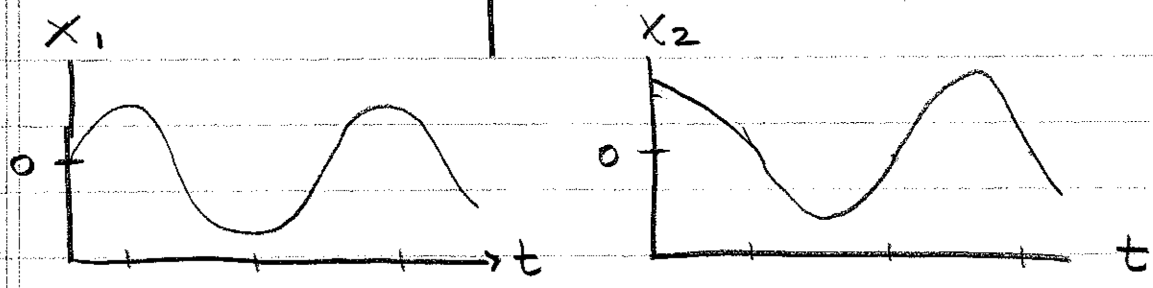
$\dot{x}_2 = -\frac{k}{m} x_1 < 0 \quad \downarrow$

$\longrightarrow + \downarrow = \searrow$

So we can start to see the closed orbit trajectories of this system that circle around the fixed point $\bar{x}^* = (0,0)$



Closed orbits are a <u>fixed oscillation</u>



would we classify this fixed point as stable or unstable? ... <u>neither</u>

So we have new kinds of fixed points in 2D systems.

BUT Before we get to that ... lets look at some other 2D vector fields.

Book looks at uncoupled 2D linear system

$\dot{x}_1 = a x_1$

$\dot{x}_2 = -x_2$

recall for $\frac{dx}{dt} = kx$; $x(0) = x_0$

$x(t) = x_0 e^{kt}$

so

$x_1(t) = x_{10} e^{at}$ → $a < 0$ exp decay, $a > 0$ exp growth

$x_2(t) = x_{20} e^{-t}$ → exponential decay

uncoupled systems are convenient because the directional components of their vector fields & solutions fall along the axis of the phase plane.

$\dot{x}_1$ component is either ⟶ or ⟵

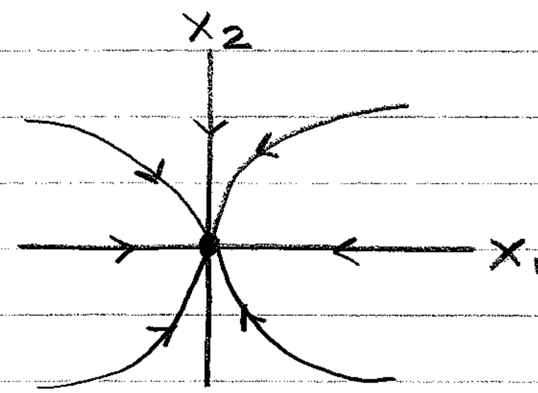
$\dot{x}_2$ component is either ↑ or ↓

so when $a < 0$ $x_1(t)$ given an exponential decay $|a|$ gives how <u>fast</u> it decays.

$a < -1$

x_1 decays faster than x_2

$\overline{x}^* = (0,0)$ → stable f.p.

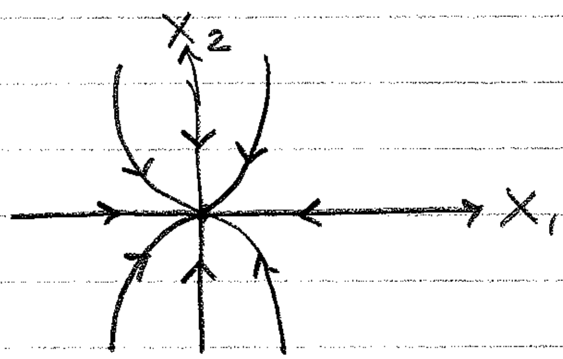


the trajectories approach the fixed point tangent to the slower direction as $t \to \infty$

if you run time backwards $t \to -\infty$ the trajectories become // with the faster direction.

so when $-1 < a < 0$

x_1 decays <u>slower</u> than x_2



how much the trajectories curve depends on how much <u>slower</u>