Sampling, Nyquist, Discrete Transforms, and all that

-or-

What Does This Box Do?

Agenda

- Sampling
  - Aliasing
  - Quantization noise
- Discrete Fourier Transform
- Lab thoughts
- Questions

Continuous vs. Discrete Time Signals

Continuous
(Defined for all values of independent variable)

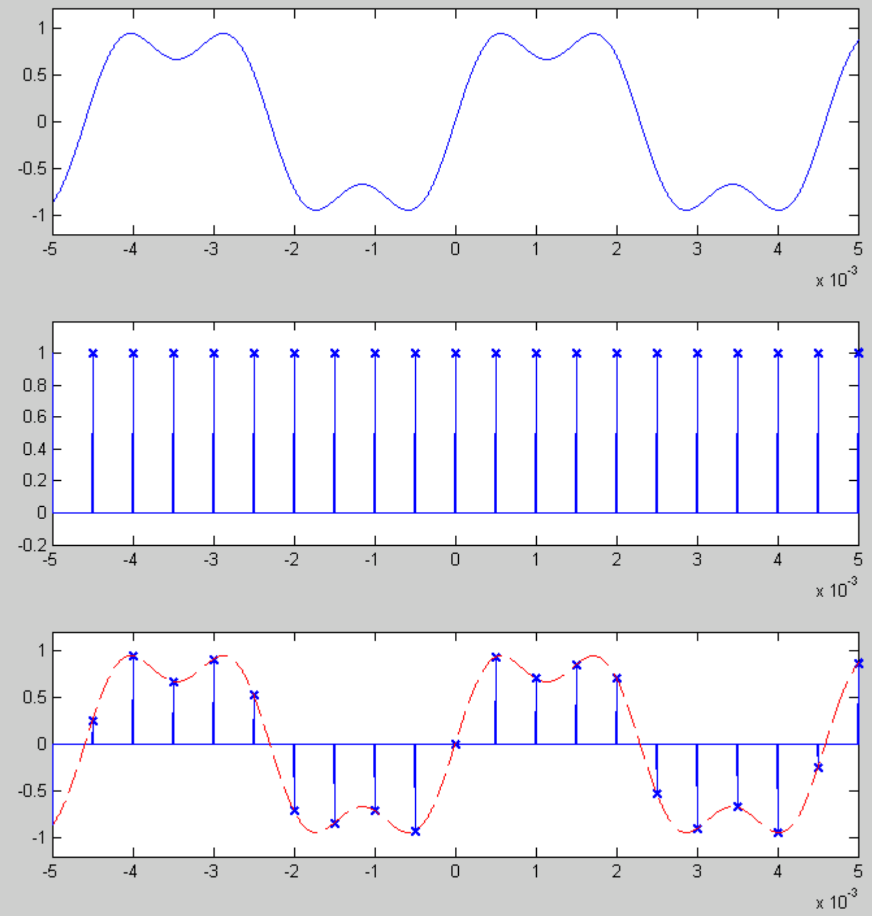
Discrete
(Defined for specific values of independent variable)

Sample

Sample

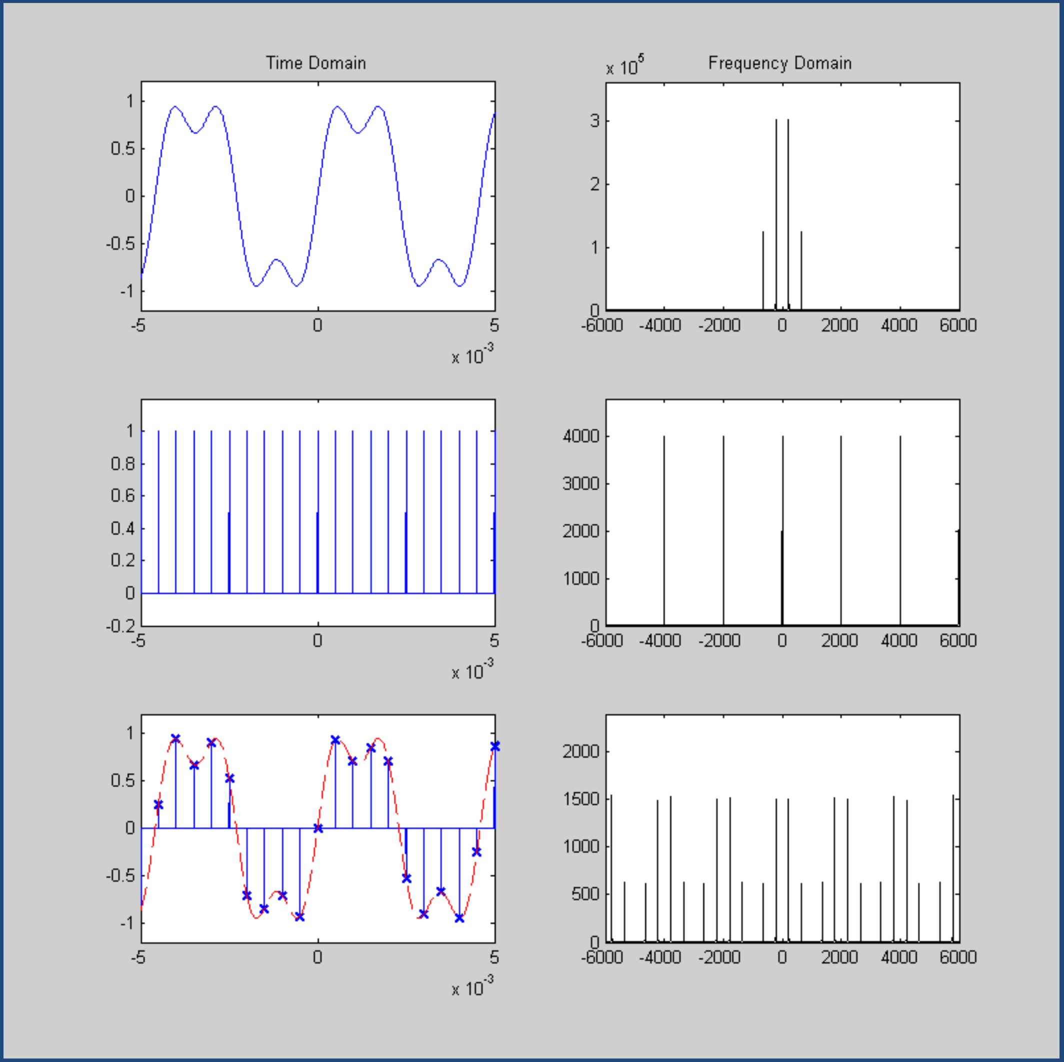
Sampling

- Record the value of a continuous function at regular intervals
  - Sampling period: *T*
  - Sampling frequency $F_s=1/T$
- Mathematically:
  - Multiply by an impulse train
    - *I(t)=1* for *t=nT*
    - *I(t)=0* otherwise
  - Almost there -- function is continuous, but = 0 outside of sample points
  - Sampled version of *v(t)*, *v[n]=v(nT)*
- Example function, *v(t)*
  - F_s=2 kHz
  - Fundamental frequency 217 Hz
  - Note: these signals are periodic and extend to t=±∞
- What does sampling do to the signal?

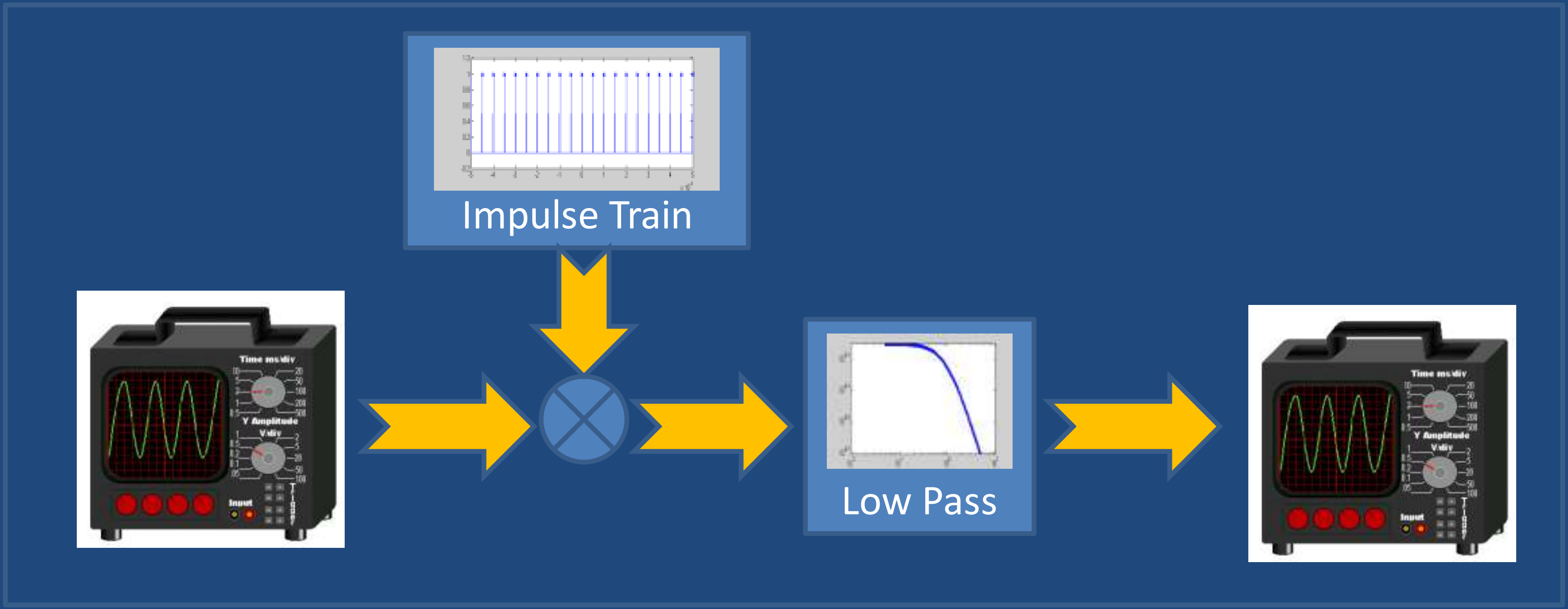


What Happens to the Signal?

- Multiplication by an impulse train causes the spectrum to be repeated at multiple frequencies
  - Convolution and multiplication are dual operations
  - More on this later (next week)
- The Fourier transform of an impulse train in time is an impulse train in frequency
  - Separated by $2\pi/T$
- Note: these signals are periodic and of infinite extent



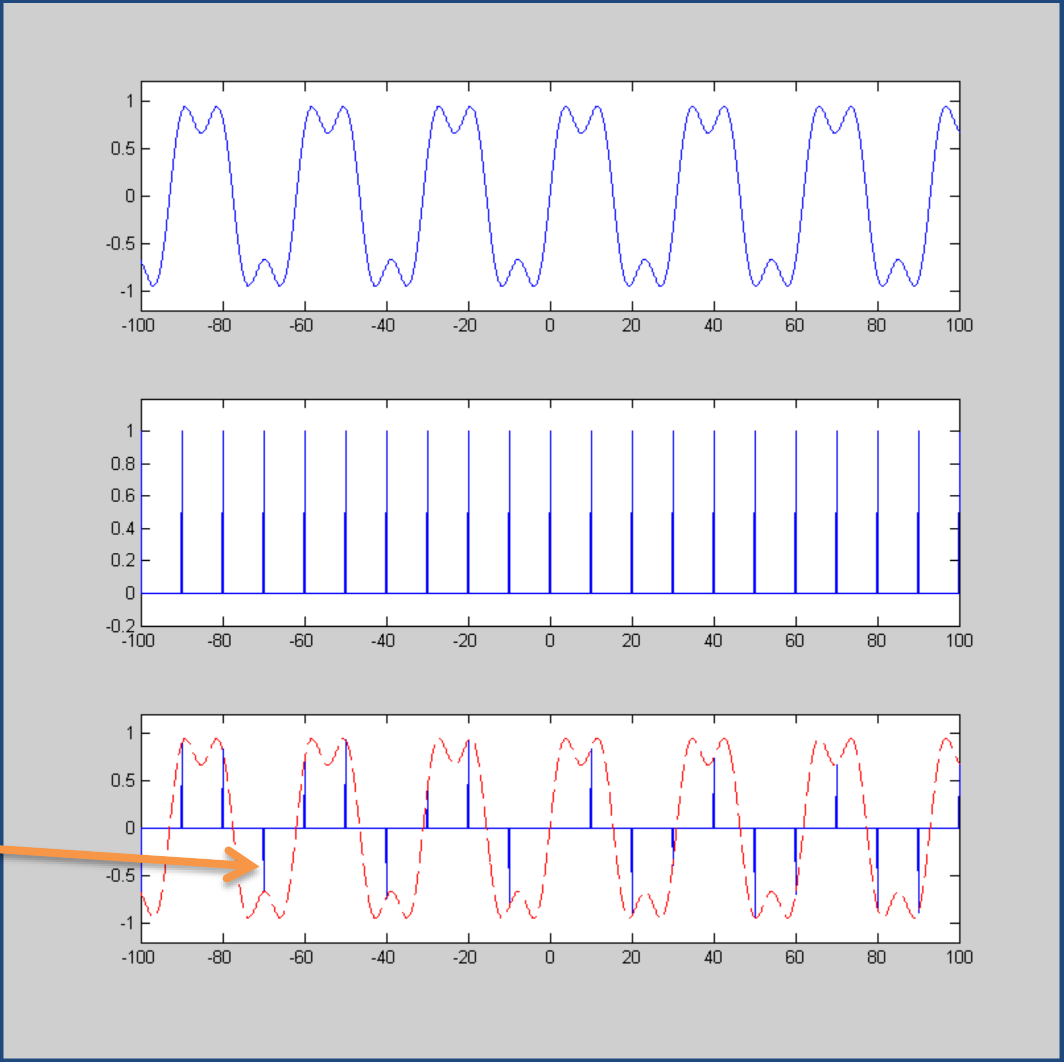
What Do You Lose?



- Signal can be reconstructed *exactly* from its samples
- How?
  - Run sampled signal through a low pass filter
- *No information is lost*
  - ***IF the sampling rate is fast enough***

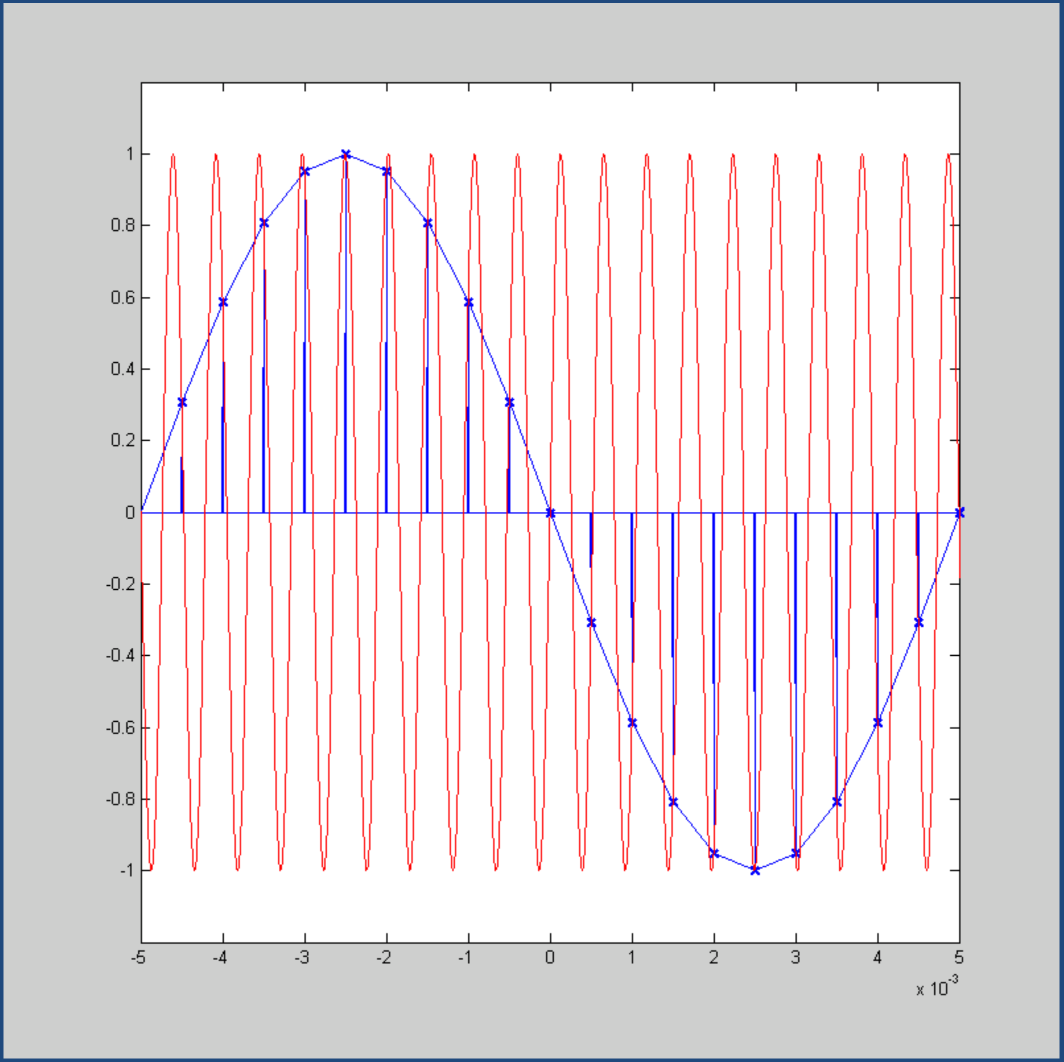
Example: Higher Frequency Waveform

- Fundamental Frequency: 651 Hz
- Third harmonic: 1953 Hz
- Seems like there are not enough sample to reconstruct the original signal
  - You can’t see the bumps with just one sample here



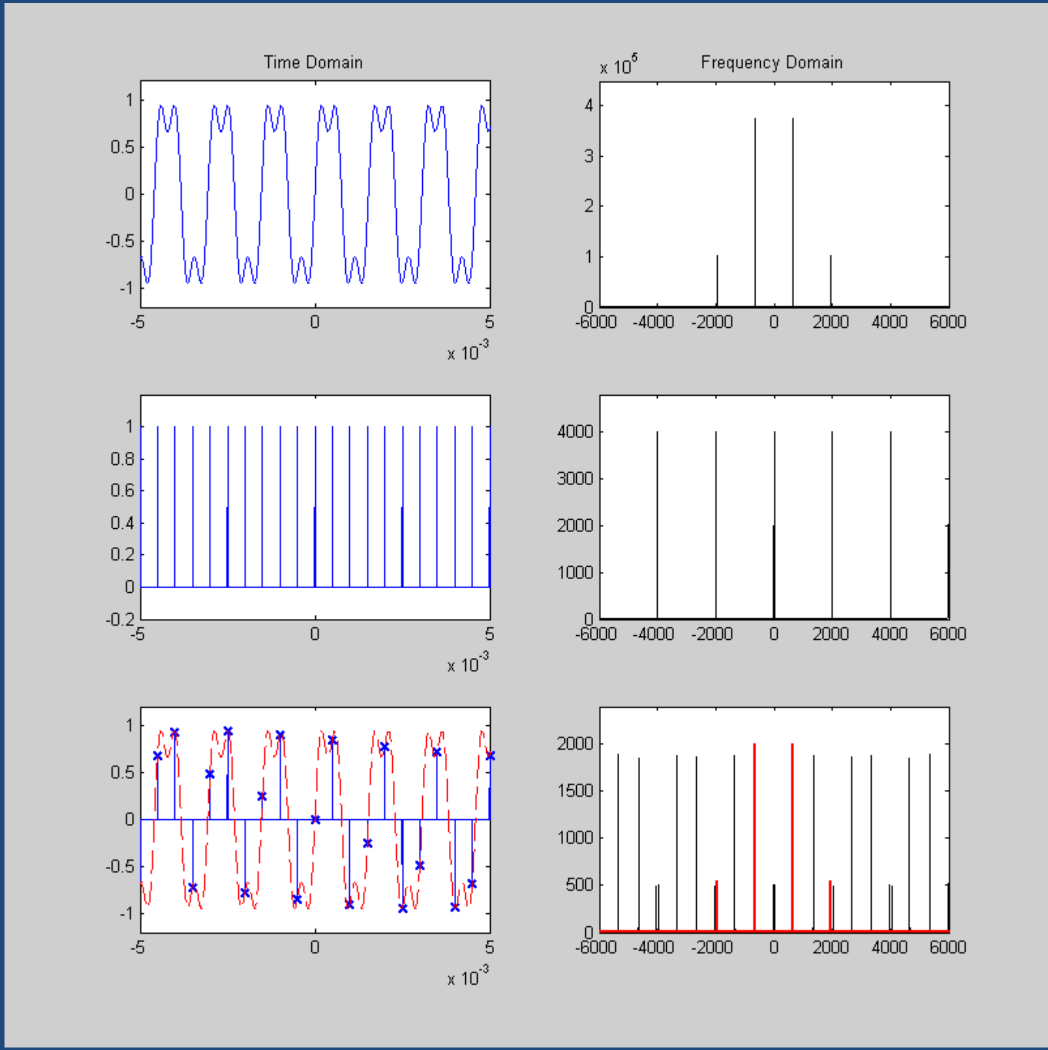
How Fast is Fast Enough?

- Nyquist frequency
  - Signal must be band-limited
  - F_s must be greater than 2x the highest frequency
    - At least two samples per cycle
- Otherwise: aliasing
  - In this example, 1900 Hz masquerades as 100 Hz (F_s = 2 kHz)
- What is aliasing?



What Happened?

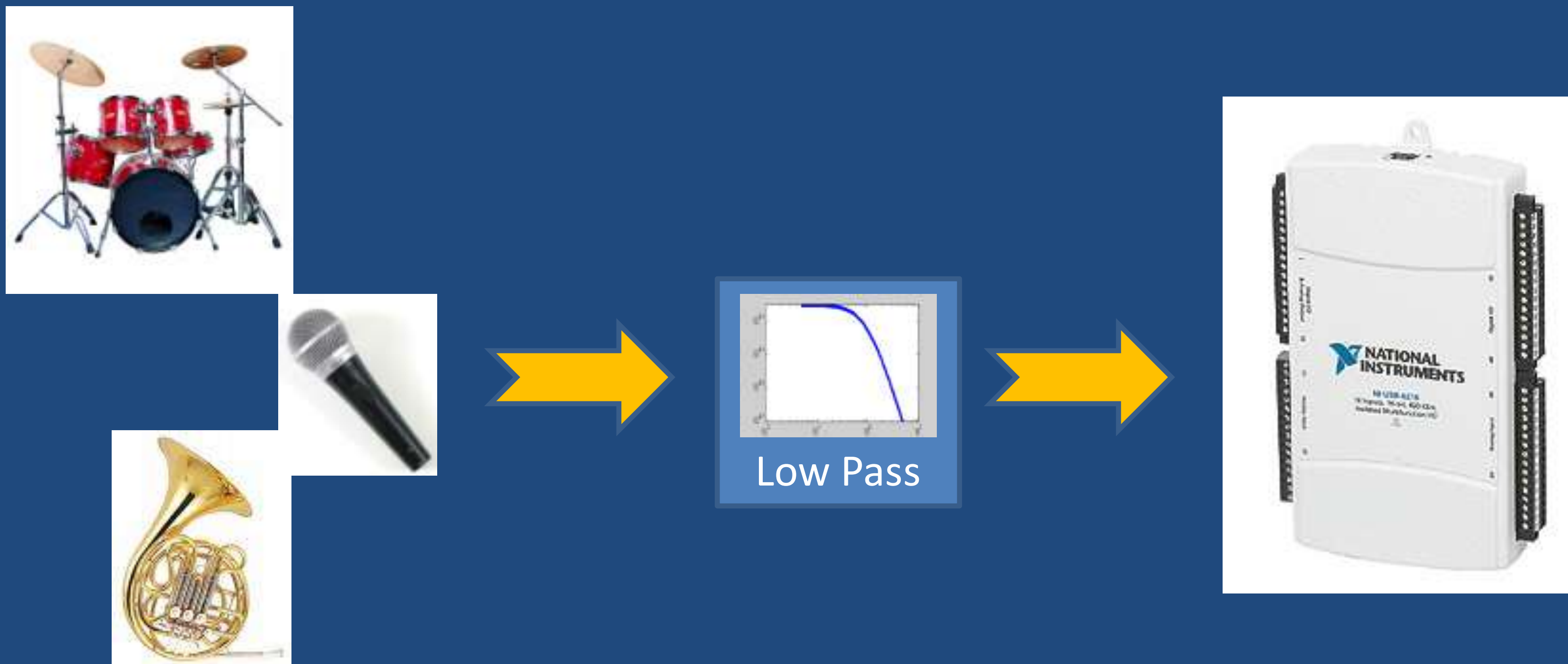
- Repeated instances of the spectrum got tangled up
  - Aliasing
- After the low pass filter, these mixed up frequencies are still present
- Frequency f ($F_s/2<f<F_s$) aliases to F_s-f
- Frequency f ($F_s<f<3F_s/2$) aliases to $f-F_s$
- And so on …



Aliasing Example: Audio

- Typical digital audio signal is band-limited to around 22 KHz with a low-pass filter and then digitized at a rate of 44.1 KHz
- Example 1: frequency sweep
  – Original waveform: band-limited, sampled at 44.1 KHz
  – Sampled at 22 KHz
  – Sampled at 2.2 KHz
- Example 2: piano notes
  – Original waveform: band-limited, sampled at 44.1 KHz
  – Sampled at 8820 Hz
  – Sampled at 4100 Hz
  – Sampled at 882 Hz
  – Sampled at 441 Hz

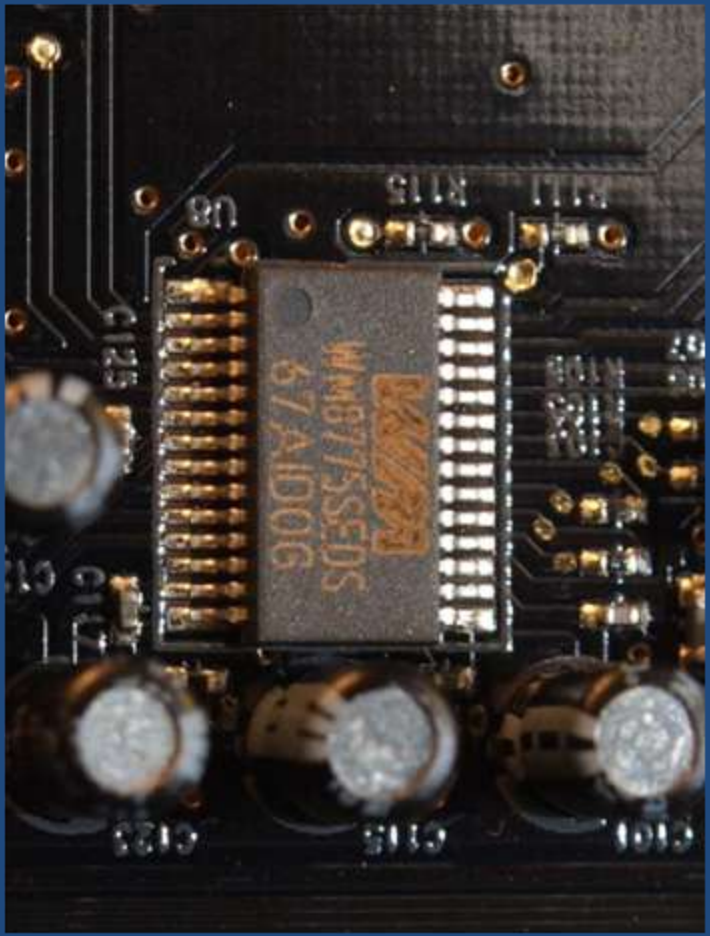
Practical Matters: Band-Limited Signals



- Most signals are not band-limited
- Analog low-pass filter before digitizing
- Even better: oversample and decimate
  - Often easier to implement good filters digitally

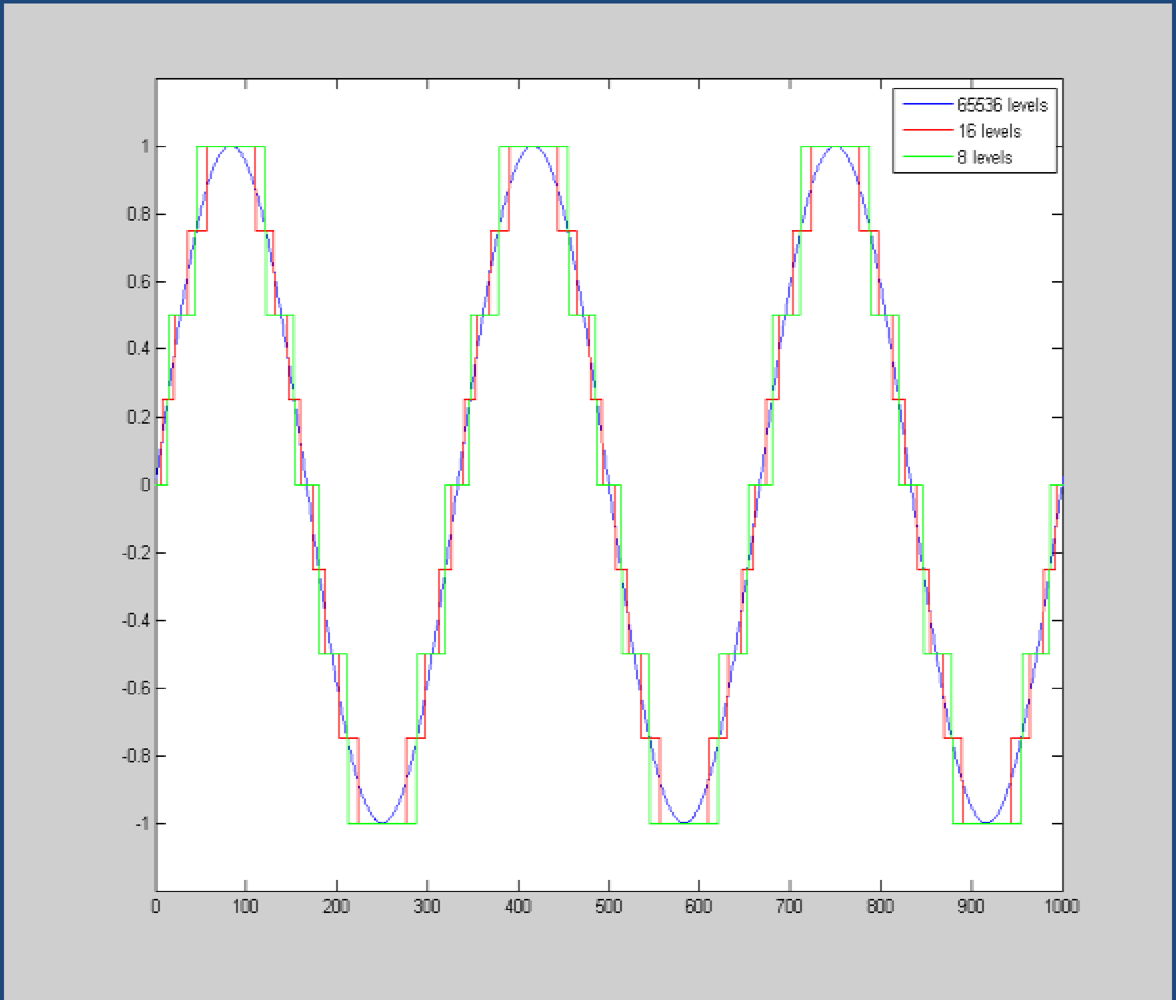
Practical Matters: A/D Converters

- ADC specifications
  - Sampling frequency
  - Number of levels
    - Resolution (bits)
  - Linearity
    - Integral nonlinearity: maximum error in any output level
    - Differential nonlinearity: maximum error between two adjacent levels



How Many Levels?

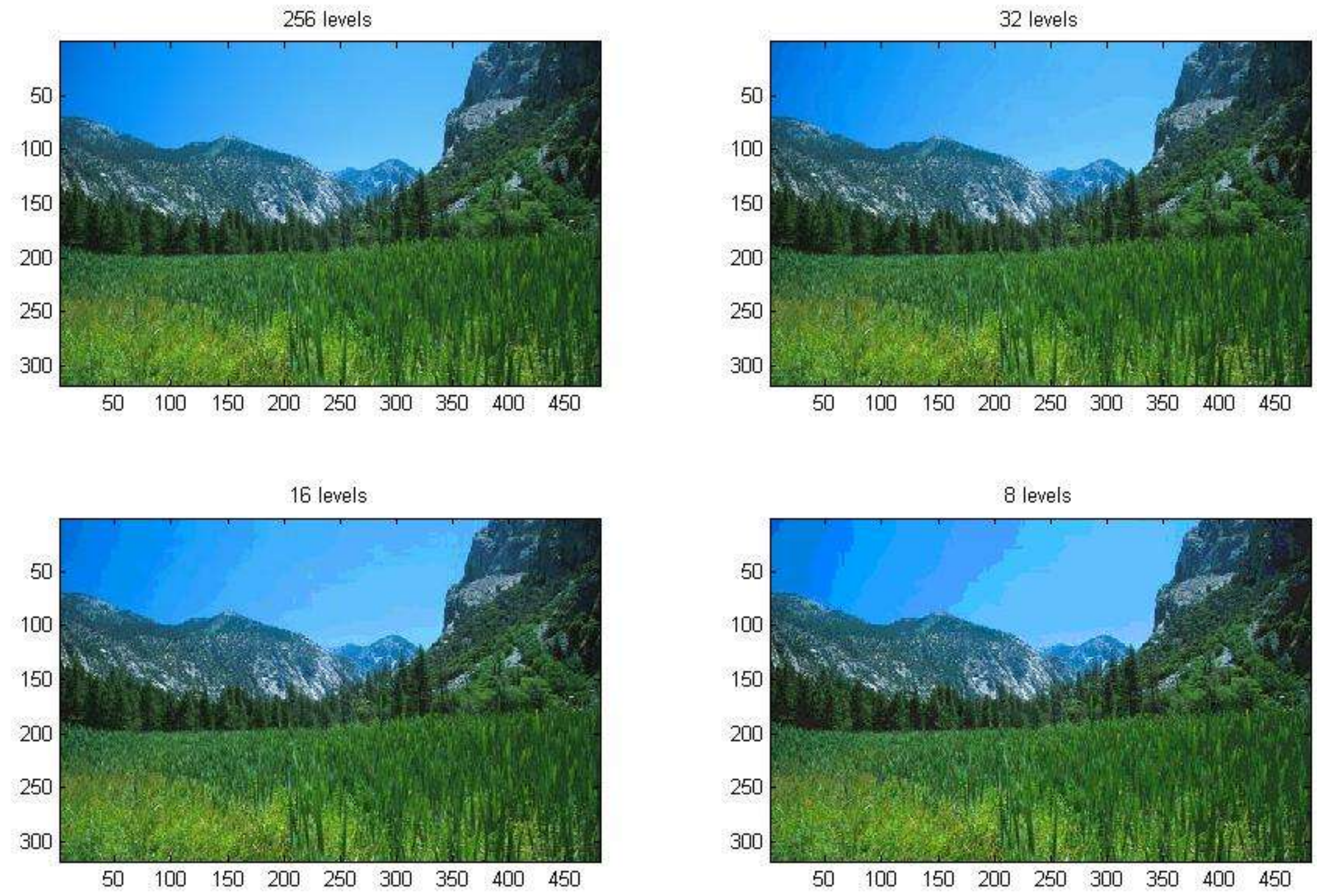
- A/D converter has a limited number of levels
  - Sampled signal is discrete in both time and value
  - Example: 16 bits
    - 65536 levels
    - Dynamic range: 96 dB
- Noise resulting from finite number of levels is called quantization noise
  - Acts like ½ LSB of random noise most of the time



Quantization Noise Demo: Sound

- Piano notes
  - 65536 levels (16 bits)
  - 16 levels (4 bits)
  - 4 levels (2 bits)
  - 2 levels (1 bit)

Quantization Noise Demo II: Banding



You have a Sampled Waveform. Now What?

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\omega n}. \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\cdot e^{i\omega n}\, d\omega$$

- Discrete Time Fourier Transform
  - Input: infinitely long sequence of (usually) real samples
  - Output: continuous, periodic function of complex frequency coefficients
  - $X(\omega)$ is periodic with period $\omega=2\pi$
- Computing this takes forever … literally
  - Useful for data described by equations, but almost never applicable to measured quantities
  - Don’t normally have the luxury of taking a measurement forever

Discrete Fourier Transform

$$\text{DFT}: \quad H[k] = \frac{1}{N} \sum_{k=0}^{N-1} h[n] e^{-ik\left(\frac{2\pi}{N}\right)n}$$

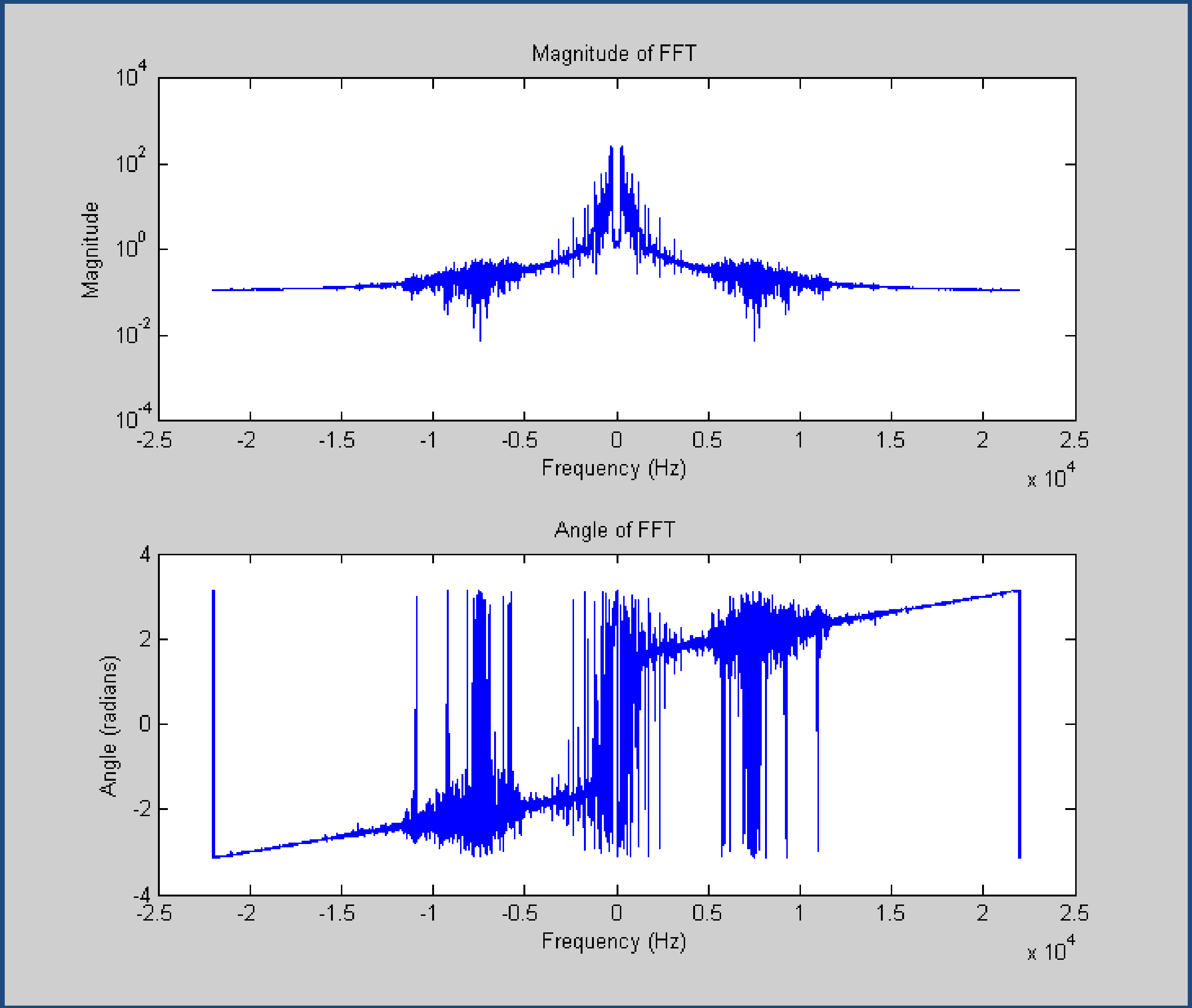
$$\text{IDFT}: \quad h[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{ik\left(\frac{2\pi}{N}\right)n}$$

- Input:
  - Finite extent – length *N*
  - Discrete in time
  - As is the case is most practical measurements
- Output:
  - *N* complex coefficients
  - Also finite length and discrete
- FFT algorithm allows efficient computation
  - O(*N* log *N*) instead of O(N^2)

DFT Example

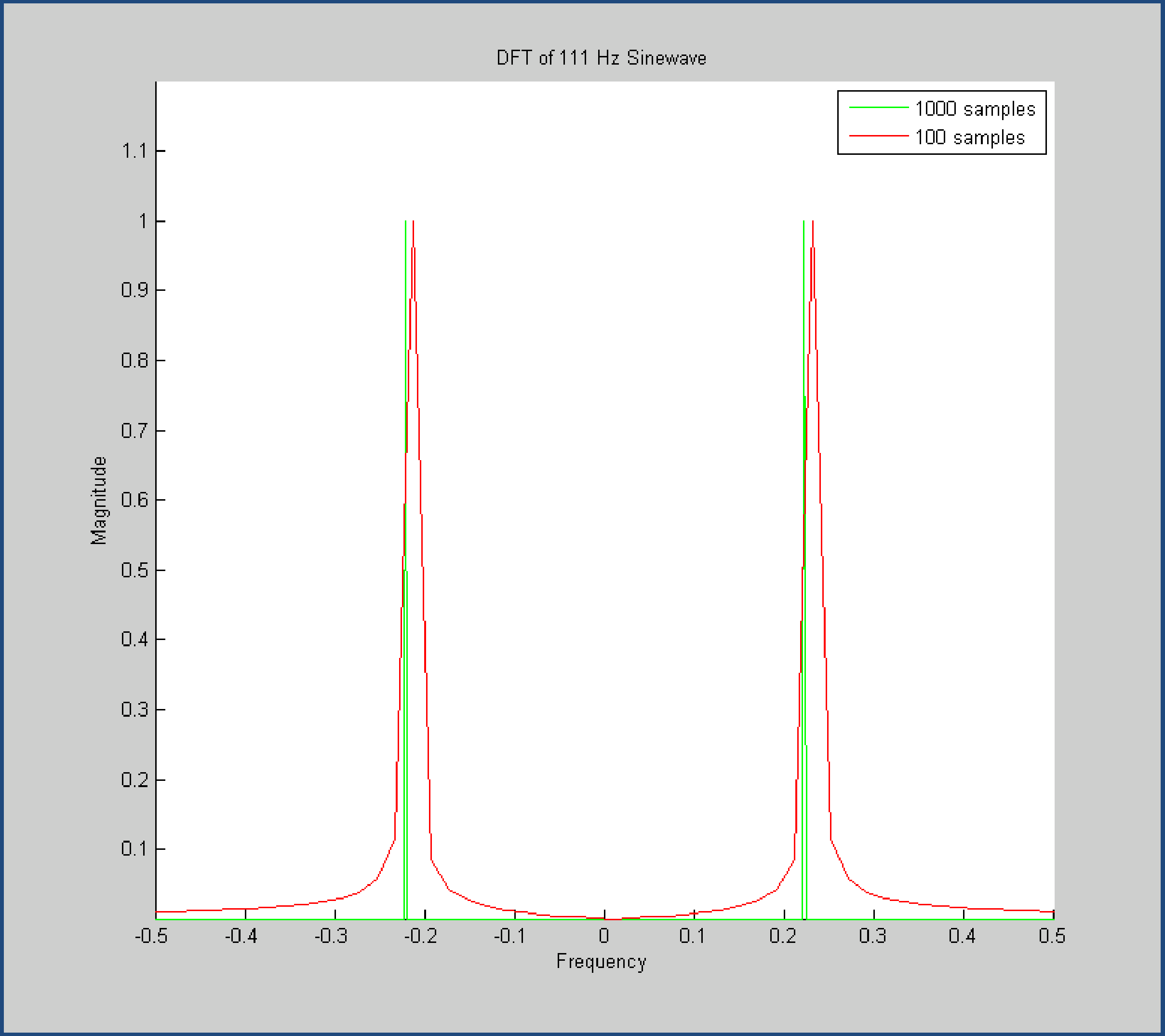
- FFT of piano sound
- Symmetric because signal is real:

$X[k]=X^*[N-k]$



Effect of Finite Signal Extent

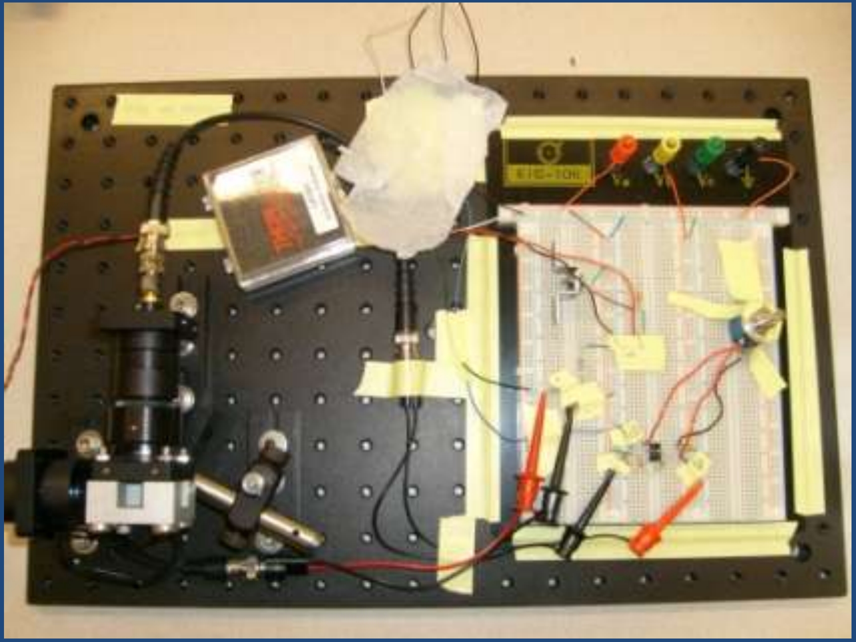
- Example: 222 Hz sinewave sampled at 1 KHz
- Spectral leakage
  - Longer signal yields more frequency bins
  - Some of the energy leaks into other bins

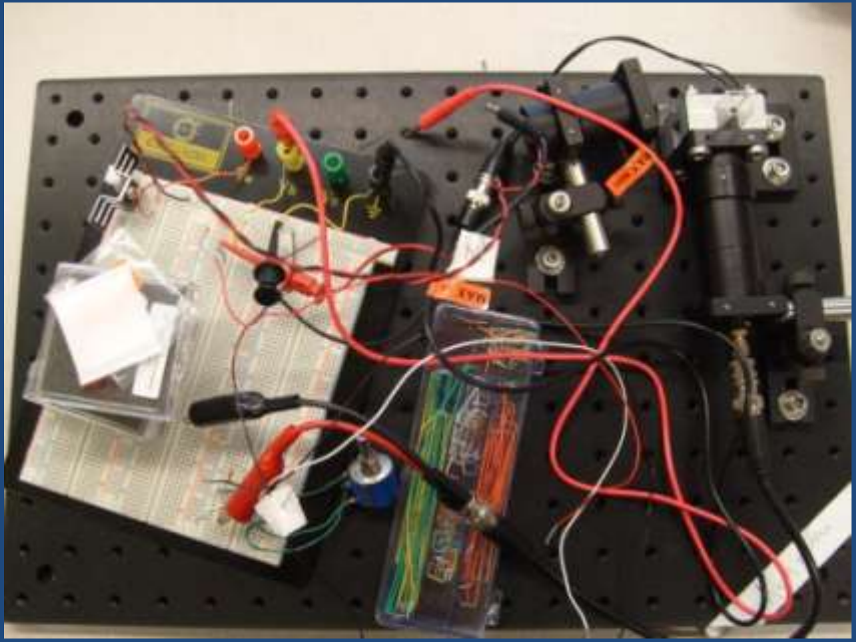


Lab Comments

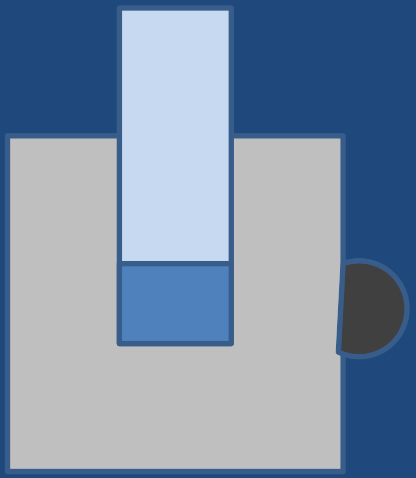
- USB-6212
  - 16 bits, 400 KHz max sample rate
  - Programmable amplifier on the input
    - Full scale sensitivity: ±10V, ±5V, ±1V, ±0.2V
    - New VI allows photodiode channel sensitivity to be adjusted
    - Possible to lower amplifier gain; compensate with DAQ
  - New VI samples at 500 Hz, decimates to 1 Hz
    - Old VI output 10 samples per second
    - 60 Hz filter in the works
- Amplifier gain
  - Offset voltage temperature coefficient: 20×10^{-6} V/°C max
  - $10^{8} \times 20 \times 10^{-6}$ V/°C = 2000V/°C max!
  - Reducing gain, increasing DAQ sensitivity may reduce drift
- Take time to make your setup robust
- You should get at least 1 good sample run this week

Neatness Counts





New heating block



- Source of error: measuring heating block temperature instead of sample
- Circuit model for heating block
- Glass vs. plastic cuvette
  - Thermal conductivity of glass >> PS
- Probe sample directly