Part 3: Steady Points of the Dynamic System

We use the dimension-less version of the system.

$$\begin{cases} \dfrac{dU}{ds} = \dfrac{U}{1+U} - \dfrac{BUV}{B_0+U} \\ \dfrac{dV}{ds} = \dfrac{CUV}{1+UV} - DV \end{cases}$$

The steady points of the Dimension-less version of the system are given by the positive solutions $(U \geq 0, V \geq 0)$ of the system

$$\begin{cases} (1) \Leftrightarrow 0 = \dfrac{U}{1+U} - \dfrac{BUV}{B_0+U} \\ (2) \Leftrightarrow 0 = \dfrac{CUV}{1+UV} - DV \end{cases}$$

Equation (1) yields $U = 0$ or $0 = \dfrac{1}{1+U} - \dfrac{BV}{B_0+U}$

Equation (2) yields $V = 0$ or $\dfrac{CU}{1+UV} = D \Leftrightarrow V = \dfrac{C}{D} - \dfrac{1}{U}$

1) First Steady Point

Plugging $V = 0$ into equation (1) yields one steady point $S_1(0,0)$

The steady point $S_1(0,0)$ is admissible whatever the values of the parameters

2) Equation for the Other Steady Points

The condition $U = 0$ is incompatible with $V = \dfrac{C}{D} - \dfrac{1}{U}$. However $0 = \dfrac{1}{1+U} - \dfrac{BV}{B_0+U}$ is.

Before studying the condition we set $R = C/D$.

The system of equations $(\otimes) \Leftrightarrow \begin{cases} BV = \dfrac{B_0+U}{1+U} \\ V = R - 1/U \end{cases}$ yield the other steady-points of the system.

We will prove that we can get up to two other points.

First one can remark from the equation $BV = \frac{B_0 + U}{1+U}$ that if we solve the system with regards to U and obtain a positive solution, then we automatically obtain a positive value of V.

We adopt this strategy, eliminate the ordinate V and seek for the positive solutions of the equation (Δ) in U.

$$(\Delta) \Leftrightarrow \qquad B\left(R - \frac{1}{U}\right) = \frac{B_0 + U}{1+U}$$

$$(\Delta) \Leftrightarrow \qquad (BR-1) - \frac{B}{U} + \frac{(1-B_0)}{1+U} = 0$$

$$(\Delta) \Leftrightarrow \qquad (BR-1)U^2 + (B(R-1) - B_0)U - B = 0$$

To obtain the positive solutions of (Δ) we need to differentiate 2 cases and study a limit case.

3) Limit Case B=1/R

The equation (Δ) degenerates into an affine equation. Its only solution is

$$U = \frac{B}{B(R-1)-B_0} = \frac{B}{1-B-B_0}$$

The steady Point is admissible if $\begin{cases} R > 1 \\ B_0 < \dfrac{R-1}{R} \end{cases}$

4) General Case B ≠ 1/R

U is given by a quadratic equation

$$P(U) = U^2 + \frac{(B(R-1) - B_0)}{(BR-1)}U - \frac{B}{(BR-1)} = 0$$

For the remaining of the study of this Dynamic System, we set $\begin{cases} s = \dfrac{(B(R-1) - B_0)}{(BR-1)} \\ p = \dfrac{B}{(BR-1)} \end{cases}$

4.1) First Case B>1/R

$$p = \frac{B}{(BR-1)} > 0$$

P(U) has one positive root $U = \frac{-s + \sqrt{s^2 + 4p}}{2}$

4.2) Second Case B<1/R

$$p = \frac{B}{(BR-1)} < 0$$

P(U) has two positive roots if and only if $\begin{cases} s < 0 \\ \Delta = s^2 + 4p \geq 0 \end{cases}$.

If the discriminant is strictly positive then the two roots are distinct.

Condition 1:

$$s = \frac{(B(R-1) - B_0)}{(BR-1)} < 0 \text{ if } B(R-1) - B_0 > 0$$

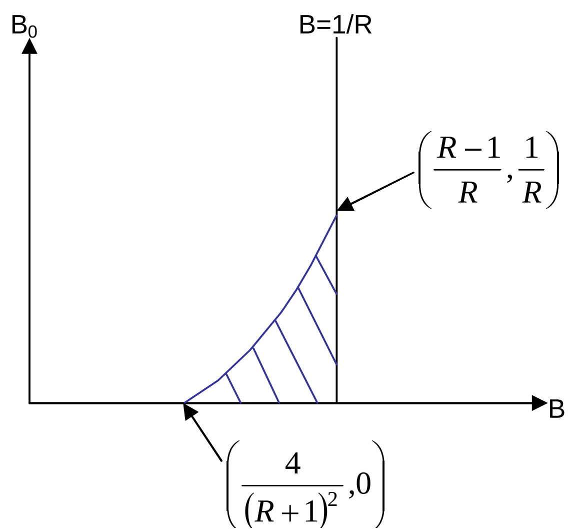
Necessary Condition: R>1

Condition 2:

$\Delta = s^2 + 4p \geq 0$ iff $s \leq -2\sqrt{-p}$

that is $B_0 \leq B(R-1) - 2\sqrt{B(1-BR)}$

One can show that for the condition to be met it takes $B \geq 4/(R+1)^2$



When the conditions are met, the roots are

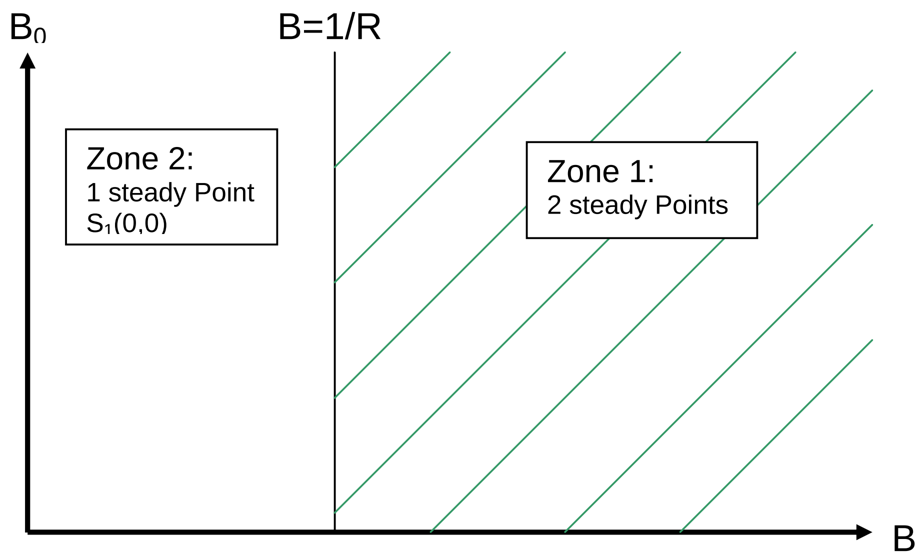
$$U_- = \frac{-s - \sqrt{s^2 + 4p}}{2}$$

and

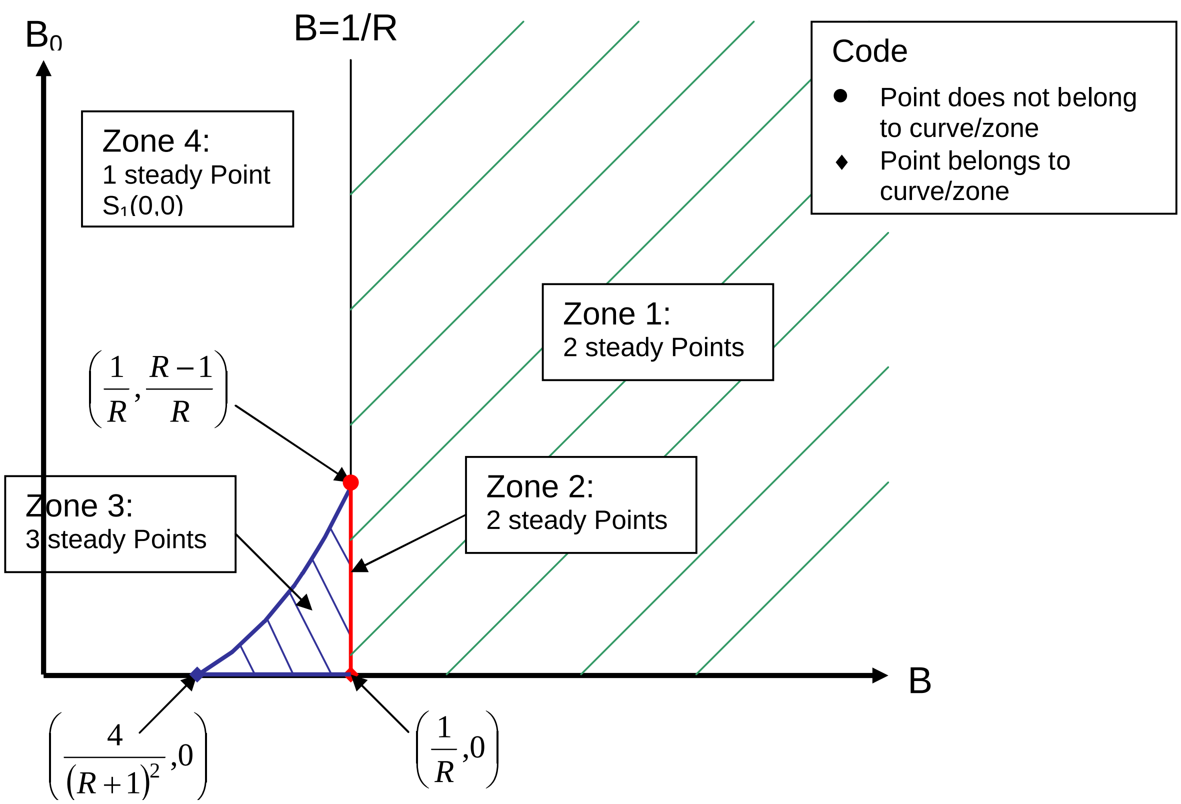
$$U_+ = \frac{-s + \sqrt{s^2 + 4p}}{2}$$

5) Bifurcation Diagrams

First Case R ≤ 1

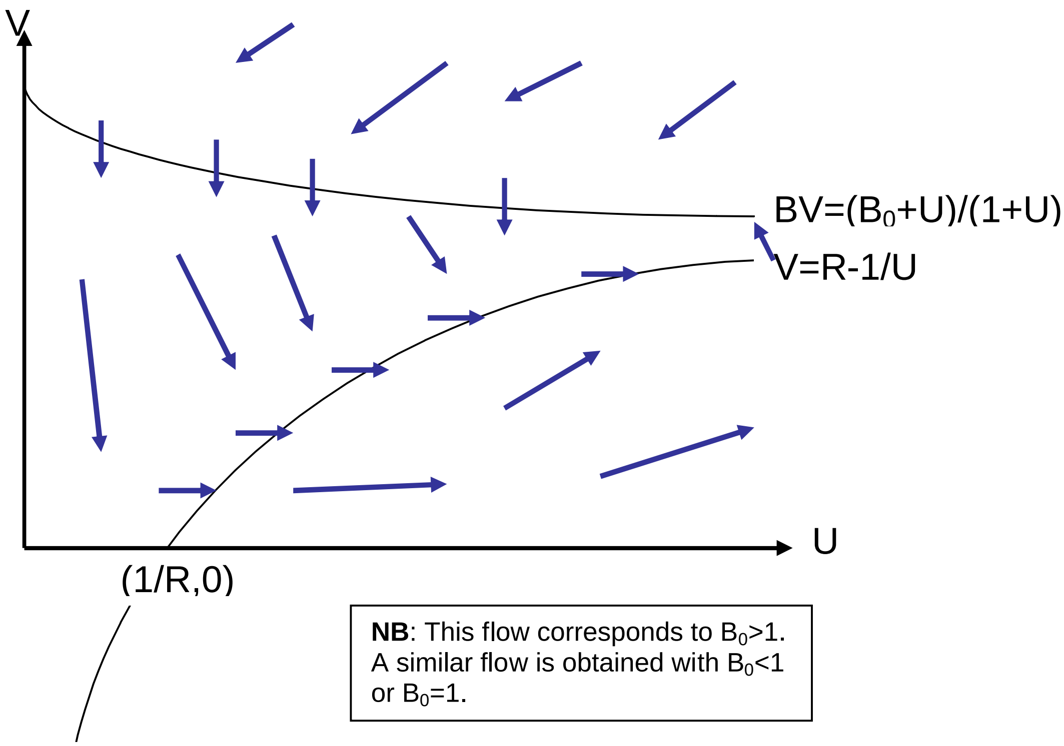


Second Case R>1



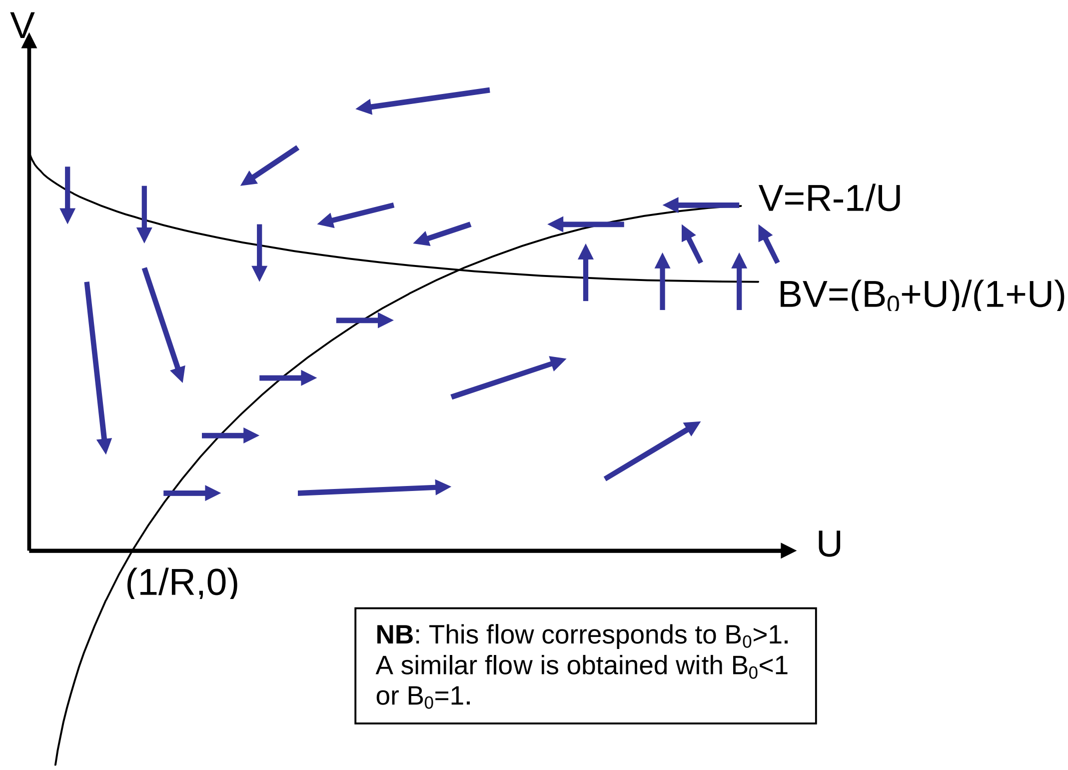
6) Complement : Allure of the Flow

Case 1: B < 1/R and Only One Steady State



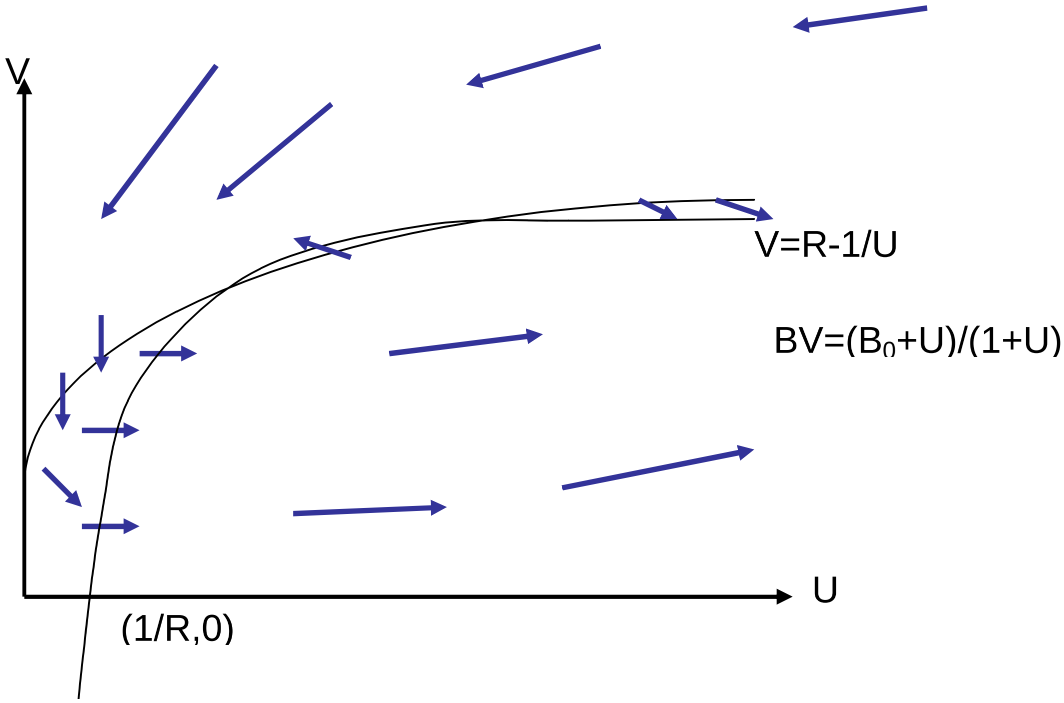
It is straightforward to see that the trajectories will not be bounded

Case 2: B > 1/R : Two Steady States



NB: This flow corresponds to $B_0>1$.
A similar flow is obtained with $B_0<1$
or $B_0=1$.

Type of Flow 3 : B < 1/R and Three Steady States



It is straightforward to see that the trajectories will not be bounded