Foundations of Analog and Digital Electronic Circuits

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significant ways. Most importantly, their branch variables do not depend algebraically upon each other. Rather, their relations involve temporal derivatives and integrals. Thus, the analysis of circuits containing capacitors and inductors involve differential equations in time. To emphasize this, we will explicitly show the time dependence of all variables in this chapter.

9.1.1 Capacitors

To understand the behavior of a capacitor, and to illustrate the manner in which a lumped model can be developed for it, consider the idealized two-terminal linear capacitor shown in Figure 9.11. In this capacitor each terminal is connected to a conducting plate. The two plates are parallel and are separated by a gap of length l. Their area of overlap is A. Note that these dimensions will be functions of time if the geometry of the capacitor varies. The gap is filled with an insulating linear dielectric having permittivity ϵ.

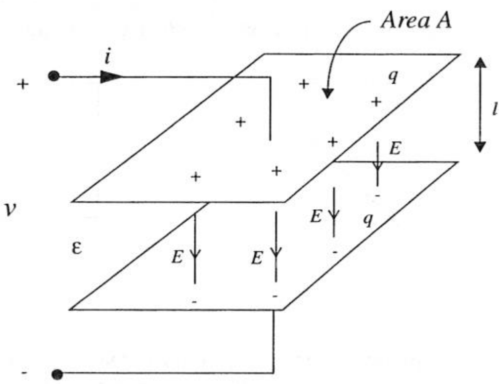


Figure 9.11: An idealized parallel-plate capacitor

As current enters the positive terminal of the capacitor it transports the electric charge q onto the corresponding plate; the units of charge is the Coulomb [C]. Simultaneously, an identical current exits the negative terminal and transports an equal charge off the other plate. Thus, although charge is separated within the capacitor, no net charge accumulates within it, as is required for lumped circuit elements by the lumped matter discipline discussed in Chapter 1.

The charge q on the positive plate and its image charge $-q$ on the negative plate produce an electric field within the dielectric. It follows from Maxwell's Equations

and the properties of linear dielectrics that the strength E of this field is

$$E(t) = \frac{q(t)}{\epsilon A(t)} \quad , \tag{9.1}$$

and its direction points from the positive plate to the negative plate. The electric field can then be integrated across the dielectric from the positive plate to the negative plate to yield

$$v(t) = l(t)E(t) \quad . \tag{9.2}$$

Combining Equations 9.1 and 9.2 then results in

$$q(t) = \frac{\epsilon A(t)}{l(t)} v(t) \tag{9.3}$$

We define

$$C(t) = \frac{\epsilon A(t)}{l(t)} \tag{9.4}$$

where C is the capacitance of the capacitor having the units of Coulombs/Volt, or Farads [F]. Substituting for the capacitance in Equation 9.3, we get

$$q(t) = C(t)v(t) \tag{9.5}$$

In contrast to the resistor, which exhibits an algebraic relation between its branch current and voltage, the capacitor does not. Rather, it exhibits an algebraic relation between its branch voltage and its stored charge. Had the dielectric not been linear, this relation would have been nonlinear. While some capacitors exhibit such nonlinear behavior, we will focus only on linear capacitors.

The rate at which charge is transported onto the positive plate of the capacitor is

$$\frac{dq(t)}{dt} = i(t) \quad . \tag{9.6}$$

From Equation 9.6 we see that the Ampere is equivalent to a Coulomb/second. Equation 9.6 can be combined with Equation 9.5 to yield

$$i(t) = \frac{d(C(t)v(t))}{dt} \tag{9.7}$$

which is the element law for an ideal linear capacitor. Unless stated otherwise, we will assume in this text that capacitors are both linear and time-invariant. For linear, time-invariant capacitors, Equations 9.5 and 9.7 reduce to

$$q(t) = Cv(t) \tag{9.8}$$

$$i(t) = C\frac{dv(t)}{dt} \quad , \tag{9.9}$$

respectively, *with the latter being the element law for a linear time-invariant capacitor.*[1]

The symbol for an ideal linear capacitor is shown in Figure 9.12. It is chosen to represent the parallel-plate capacitor shown in Figure 9.11. Also shown in the figure is a graph of the relation between the branch voltage and stored charge of the capacitor.

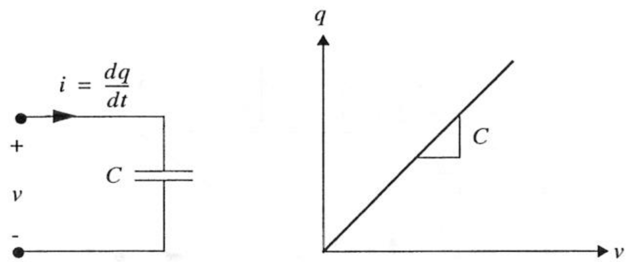


Figure 9.12: The symbol and voltage-charge relation for the ideal linear capacitor. The element law for the capacitor is $i = C\frac{dv}{dt}$

One of the important properties of a capacitor is its memory property. In fact, it is this property that allows the capacitor to be the primary memory element in all integrated circuits. To see this property, we integrate Equation 9.6 to produce

$$q(t) = \int_{-\infty}^{t} i(t)dt \tag{9.11}$$

or, with the substitution of Equation 9.8, to produce

[1]Although we will focus primarily on linear, time-invariant capacitors in this text, we note that some interesting transducers such as electric microphones and speakers, and other electric sensors and actuators, are appropriately modeled with time-varying capacitors. Similarly, most capacitors used in electronic equipment (paper, mica, ceramic, etc.) are linear, but often vary a small amount with temperature (a part of 10^4 per degree centigrade). But many are nonlinear. The charge associated with a reverse-biased semiconductor diode, for example, varies as the $2/3$ power of voltage, because the distance d, the effective width of the space-charge layer, is a function of voltage

$$q = K\left(\psi_o^{2/3} - (\psi_o - v)^{2/3}\right) \tag{9.10}$$

when ψ_o, the contact potential, is a few tenths of a volt. From the above equations the capacitance of the reverse-biased diode varies as $v^{-1/3}$.

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt \quad . \tag{9.12}$$

Equation 9.12 shows that the branch voltage of a capacitor depends on the entire past history of its branch current which is the essence of memory. This is in marked contrast to a resistor (either linear or nonlinear) which exhibits no such memory property.

At first glance, it might appear that it is necessary to know the entire history of the current i in detail in order to carry out the integrals in Equations 9.11 and 9.12. This is actually not the case. For example, consider rewriting Equation 9.11 as

$$\begin{aligned} q(t_2) &= \int_{-\infty}^{t_2} i(t)dt \\ &= \int_{t_1}^{t_2} i(t)dt + \int_{-\infty}^{t_1} i(t)dt \\ &= \int_{t_1}^{t_2} i(t)dt + q(t_1) \quad . \end{aligned} \tag{9.13}$$

The latter equality shows that $q(t_1)$ perfectly summarizes, or memorizes, the entire accumulated history of $i(t)$ for $t \leq t_1$. Thus, if $q(t_1)$ is known, it is necessary and sufficient to know i only over the interval $t_1 \leq t \leq t_2$ in order to determine $q(t_2)$. For this reason, q is referred to as the *state* of the capacitor. For linear time-invariant capacitors, v can also easily serve as a state because v is proportionally related to q through the constant C. Accordingly, rewriting Equation 9.12 as

$$\begin{aligned} v(t_2) &= \frac{1}{C}\int_{-\infty}^{t_2} i(t)dt \\ &= \frac{1}{C}\int_{t_1}^{t_2} i(t)dt + \frac{1}{C}\int_{-\infty}^{t_1} i(t)dt \\ &= \frac{1}{C}\int_{t_1}^{t_2} i(t)dt + v(t_1) \quad . \end{aligned} \tag{9.14}$$

Thus, we see that $v(t_1)$ also memorizes the entire accumulated history of $i(t)$ for $t \leq t_1$ and can serve as the state of the capacitor.

Associated with the ability to exhibit memory is the property of energy storage, which is often exploited by circuits that process energy. To determine the electric energy w_E stored in a capacitor, we recognize that the power iv is the rate at which energy is delivered to the capacitor through its port. Thus,

$$\frac{dw_E(t)}{dt} = i(t)v(t) \quad . \tag{9.15}$$

Next, substitute for i using Equation 9.6, cancel the time differentials, and omit the parametric time dependence to obtain

$$dw_E = vdq \quad . \tag{9.16}$$

Equation 9.16 is a statement of incremental energy storage within the capacitor. It states that the transport of the incremental charge dq from the negative plate of the capacitor to the top plate across the electric potential difference v stores the incremental energy dw_E within the capacitor. To obtain the total stored electric energy, we must integrate Equation 9.16 with v treated as a function of q. This yields

$$w_E = \int_0^q v(x)dx \qquad (9.17)$$

where x is a dummy variable of integration. Finally, substitution of Equation 9.8 and integration yields

$$\text{Stored Energy} = w_E(t) = \frac{q^2(t)}{2C} = \frac{Cv(t)^2}{2} \qquad (9.18)$$

as the electric energy stored in a capacitor. The units of energy is the Joule [J], or Watt-second. *Unlike a resistor, a capacitor stores energy rather than dissipates it.*

Capacitors come in an enormous range of values. For example, two pieces of insulated wire about an inch long, when twisted together will have a capacitance of about 1 picofarad (10^{-12} farads). A low-voltage power supply capacitor an inch in diameter and a few inches long could have a capacitance of 100,000 microfarads (0.1 farad. 1 microfarad, abbreviated as μF, is 10^{-6} farads).

A real capacitor can exhibit richer behavior than that described above. For example, leakage current can flow through its dielectric. The practical significance of dielectric leakage is that eventually the charge stored on a capacitor can leak off. Thus, eventually a real capacitor will lose its memory. Fortunately, capacitors can be made with very low leakage (in other words, with very high resistance) in which case they are excellent long-term memory devices. However, if the dielectric leakage is large enough to be significant, then it can be modeled with a resistor in parallel with the capacitor.

Other nonidealities include the distributed series resistance, and even series inductance, that arises in foil-wound capacitors in particular. These characteristics limit the power handling capability of a real capacitor, and the frequency range over which a real capacitor behaves like an ideal capacitor. They can often be explicitly modeled with a single series resistor and inductor, respectively.

Example 9.1 Parallel Plate Capacitor

Suppose the parallel plate capacitor in Figure 9.11 is 1 m square, has a gap separation of 1 μm, and is filled with a dielectric having permittivity of $2\epsilon_o$, where $\epsilon_o \approx 8.854 \times 10^{-12}$ F/m is the permittivity of free space. What is its capacitance? How much charge and energy does it store if its terminal voltage is 100 V?

The capacitance is determined from Equation 9.4 with $\epsilon = 1.8 \times 10^{-11}$ F/m, $A = 1$ m^2 and $l = 10^{-6}$ m. It is 18 μF. The charge is determined from Equation 9.8 with $v = 100$ V. It is 1.8 mC. Finally, the stored energy is determined from Equation 9.18. It is 90 mJ.

9.1.2 Inductors

As we saw in Section 9.1.1, from the perspective of modeling electrical systems, the capacitor is a circuit element to model the effect of electric fields. Correspondingly, the *inductor* models the effect of magnetic fields. To understand the behavior of an inductor, and to illustrate the manner in which a lumped model can be developed for it, consider the idealized two-terminal linear inductor shown in Figure 9.13. In this inductor a coil with a terminal on each end is wound with N turns around a toroidal core made from an insulator having magnetic permeability μ. The length around the core is l and its cross-sectional area is A. Note that these dimensions will be functions of time if the geometry of the inductor varies.

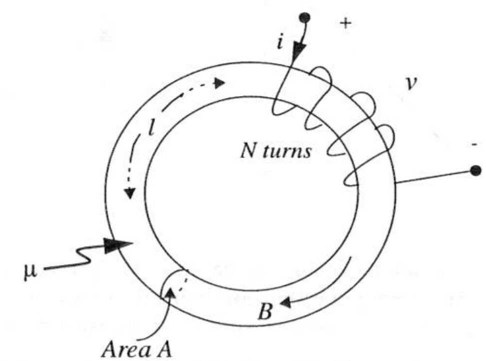


Figure 9.13: An idealized toroidal inductor

The current in the coil produces a magnetic flux in the inductor. Ideally, this magnetic flux does not stray significantly from the core, so that the flux outside the element is negligible. Thus the inductor can be treated as a lumped circuit element that satisfies the lumped matter discipline discussed in Chapter 1. From Maxwell's Equations and the properties of permeable materials, the density B of the flux is

$$B(t) = \frac{\mu N i(t)}{l(t)} , \qquad (9.19)$$

9.2.1 Capacitors

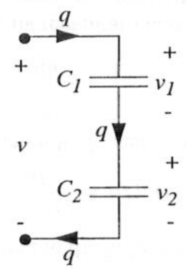


Figure 9.15: Two capacitors in series

Consider first the series combination of two capacitors as shown in Figure 9.15; we will assume here that the two capacitors were uncharged at the time of their connection. Since the two capacitors share a common current, it follows from Equation 9.11 that they store a common charge q, as shown in the figure. Thus, following Equation 9.8,

$$q(t) = C_1 v_1(t) = C_2 v_2(t) \quad . \tag{9.37}$$

Next, using KVL we observe that

$$v(t) = v_1(t) + v_2(t) \quad . \tag{9.38}$$

Finally, since the effective capacitance C of the two series capacitors is q/v, it follows that

$$\frac{1}{C} = \frac{v(t)}{q(t)} = \frac{1}{C_1} + \frac{1}{C_2} \quad ,$$

or,

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{9.39}$$

where the second equality results from the substitution of Equation 9.38 and then Equation 9.37. Thus, we see that the reciprocal capacitances of capacitors in series add. This is consistent with the physical derivation of capacitance in Equation 9.4 since placing capacitors in series essentially increases their combined gap length.

Now consider the parallel combination of two capacitors as shown in Figure 9.16. Since the two capacitors share a common voltage v, it follows from 9.8 that

$$v(t) = \frac{q_1(t)}{C_1} = \frac{q_2(t)}{C_2} \quad . \tag{9.40}$$

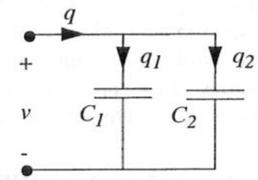


Figure 9.16: Two capacitors in parallel

Next, using KCL and Equation 9.11 we observe that

$$q(t) = q_1(t) + q_2(t) \quad . \tag{9.41}$$

Finally, since the effective capacitance C of the two parallel capacitors is q/v, it follows that

$$C = \frac{q(t)}{v(t)} = C_1 + C_2 \tag{9.42}$$

where the second equality results from the substitution of Equation 9.41 and then Equation 9.40. Thus, we see that the capacitances of capacitors in parallel add. This is consistent with the physical derivation of capacitance in Equation 9.4 since placing capacitors in parallel essentially increases their combined cross sectional area.

Example 9.3 Capacitor Combinations

What equivalent capacitors can be made by combining up to three 1-μF capacitors in series and/or in parallel?

Figure 9.17 shows the possible capacitor combinations which use up to three capacitors. To determine their equivalent capacitances, use the series combination result from Equation 9.39 and/or the parallel combination result from Equation 9.42. This yields the equivalent capacitances of: (A) 1 μF, (B) 2 μF, (C) 0.5 μF, (D) 3 μF, (E) 1.5 μF, (F) 0.667 μF, and (G) 0.333 μF.

9.2.2 Inductors

Consider the series combination of two inductors as shown in Figure 9.18; we will assume here that neither inductor carried a current at the time of their connection. Since the two inductors share a common current i, it follows from Equation 9.27 that

$$i(t) = \frac{\lambda_1(t)}{L_1} = \frac{\lambda_2(t)}{L_2} \quad . \tag{9.43}$$

Chapter 10

First-order Transients in Linear Electrical Networks

As illustrated in Chapter 9, capacitances and inductances impact circuit behavior. The effect of capacitances and inductances is so acute in high-speed digital circuits, for example, that our simple digital abstractions developed in Chapter 6 based on a static discipline become insufficient for signals that undergo transitions. Therefore, understanding the behavior of circuits containing capacitors and inductors is important. In particular, this chapter will augment our digital abstraction with the concept of delay to include the effects of capacitors and inductors.

Looked at positively, because they can store energy, capacitors and inductors display the memory property, and offer signal-processing possibilities not available in circuits containing only resistors. Apply a square wave voltage to a multi-resistor linear circuit, and all of the voltages and currents in the network will have the same square-wave shape. But include one capacitor in the circuit and very different waveforms will appear - sections of exponentials, spikes, sawtooth waves. Figure 10.1 shows an example of such waveforms for the two-inverter system of Figure 9.1 in Chapter 9. The linear analysis techniques already developed — node equations, superposition, etc., are adequate for finding appropriate network equations to analyze these kinds of circuits. However, the formulations turn out to be *differential equations* rather than algebraic equations, so additional skills are needed to complete the analyses.

This chapter will discuss systems containing a single storage element, namely, a single capacitor or a single inductor. Such systems are described by simple, first-order differential equations. Chapter 12 will discuss systems containing two storage elements. Systems with two storage elements are described by second order differential equations.[1] Higher order systems are also possible, and are discussed briefly in Chapter 12.

[1]However, a circuit with two storage elements that can be replaced by a single equivalent storage element remains a first-order circuit. For example, a pair of capacitors in parallel can be replaced with a single capacitor whose capacitance is the sum of the two capacitances.

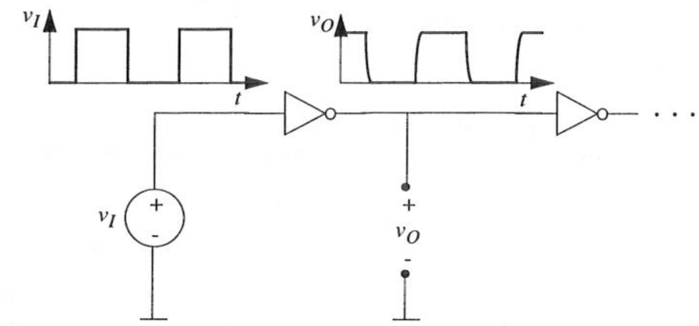


Figure 10.1: Observed response of the first inverter to a square wave input

This chapter will start by analyzing simple circuits containing one capacitor, one resistor and possibly a source. We will then analyze circuits containing one inductor and one resistor. The two inverter circuit of Figure 10.1 is examined in detail in Section 10.4.

10.1 Analysis of RC Circuits

Let us illustrate first-order systems with a few primitive examples containing a resistor, a capacitor and a source. We first analyze a current source driving the so called parallel RC circuit.

10.1.1 Parallel RC Circuit, Step Input

Shown in Figure 10.2a is a simple source-resistor-capacitor circuit. On the basis of the Thévenin and Norton equivalence discussion in Section 3.6.1, this circuit could result from a Norton transformation applied to a more complicated circuit containing many sources and resistors, and one capacitor, as suggested in Figure 10.3. Let us assume we wish to find the capacitor voltage v_C. We will use the node method described in Chapter 3 to do so. As shown in Figure 10.2a, we take the bottom node as ground, which leaves us with one unknown node voltage corresponding to the top node. The voltage at the top node is the same as the voltage across the capacitor, and so we will proceed to work with v_C as our unknown. Next, according to Step 3 of the node method, we write KCL for the top node in Figure 10.2a, substituting the constituent relation for a capacitor from Equation 9.9,

$$i(t) = \frac{v_C}{R} + C\frac{dv_C}{dt} \tag{10.1}$$

Or, rewriting,

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{i(t)}{C} \tag{10.2}$$

As promised, the problem can be formulated in one line. But to find $v_C(t)$, we must solve a nonhomogeneous, linear first-order ordinary differential equation with constant coefficients. This is not a difficult task, but one that must be done systematically using any method of solving differential equations.

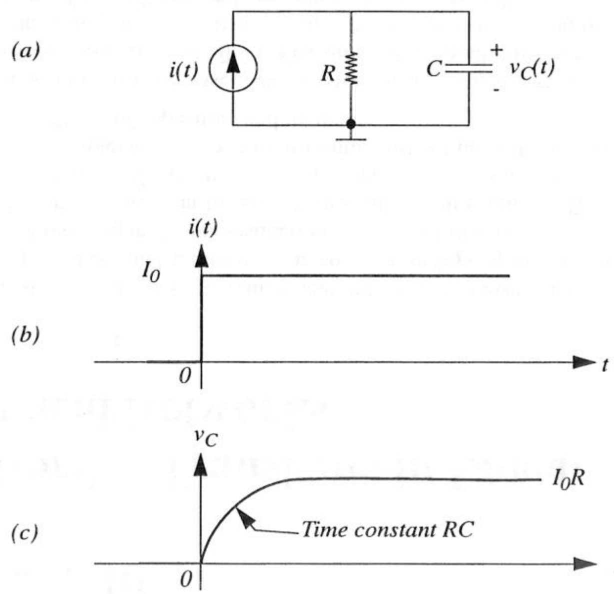


Figure 10.2: Capacitor charging transient

To solve this equation, we will use the *method of homogeneous and particular solutions* because this method can be readily extended to higher order equations. As a review, the method of homogeneous and particular solutions arises from a fundamental theorem of differential equations. The method states that the solution to the nonhomogeneous differential equation can be obtained by summing together the homogeneous

solution and the particular solution. More specifically, let $v_{CH}(t)$ be any solution to the homogeneous differential equation

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = 0 \tag{10.3}$$

associated with our nonhomogeneous differential equation 10.2. The homogeneous equation is derived from the original nonhomogeneous equation by setting the driving function, $i(t)$ in this case, to zero. Further, let $v_{CP}(t)$ be *any* solution to Equation 10.2. Then, the sum of the two solutions,

$$v_C(t) = v_{CH}(t) + v_{CP}(t)$$

is a general solution or a total solution to Equation 10.2. $v_{CH}(t)$ is called the homogeneous solution and $v_{CP}(t)$ is called the particular solution. When dealing with circuit responses, the homogeneous solution is also called the *natural response* of the circuit because it depends only on the internal energy storage properties of the circuit and not on external inputs. The particular solution is also called the forced response or the forced solution because it depends on the external inputs to the circuit.

Let us now return to the business of solving Equation 10.2. To make the problem specific, assume that the current source $i(t)$ is a step function

$$i(t) = I_0 \qquad t > 0 \tag{10.4}$$

as shown in Figure 10.2b. Further, we assume for now that the voltage on the capacitor was zero before the current step was applied. In mathematical terms, this is an *initial condition*

$$v_C = 0 \qquad t < 0 \tag{10.5}$$

> As discussed earlier, the method of homogeneous and particular solutions proceeds in three steps:
>
> 1. Find the homogeneous solution v_{CH}.
> 2. Find the particular solution v_{CP}.
> 3. The total solution is then the sum of the homogeneous solution and the particular solution. Use the initial conditions to solve for the remaining constants.

The first step is to solve the homogeneous equation, formed by setting the driving function in the original differential equation to zero. Then, any method of solving homogeneous equations can be used. In this case the homogeneous equation is:

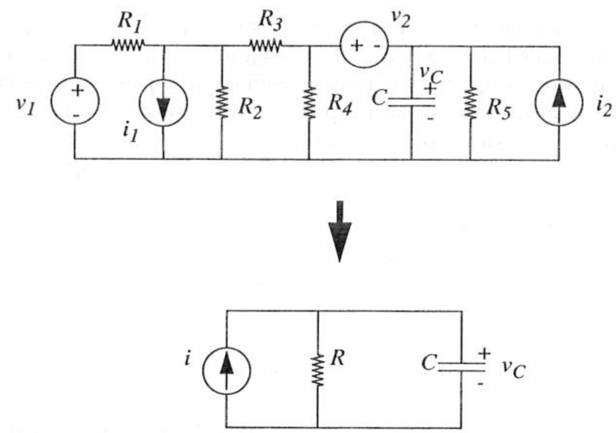


Figure 10.3: A more complicated circuit that can be transformed into the simpler circuit in Figure 10.2a by using Thévenin and Norton transformations

$$\frac{dv_{CH}}{dt} + \frac{v_{CH}}{RC} = 0 \tag{10.6}$$

We assume a solution of the form

$$v_{CH} = Ae^{st} \tag{10.7}$$

because the homogeneous solution for any linear constant-coefficient ordinary differential equation is always of this form. Now we must find values for the constants A and s. Substitution into Equation 10.6 yields

$$Ase^{st} + \frac{Ae^{st}}{RC} = 0 \tag{10.8}$$

The value for A cannot be determined from this equation, but discarding the trivial solution of $A = 0$, we find

$$\boxed{s + \frac{1}{RC} = 0} \tag{10.9}$$

because e^{st} is never zero for finite s and t, so can be factored out. Hence

$$\boxed{s = -\frac{1}{RC}} \tag{10.10}$$

Equation 10.9 is called the characteristic equation of the system, and $s = -\frac{1}{RC}$ is a root of this characteristic equation. The characteristic equation summarizes the fundamental dynamic properties of a circuit, and we will have much more to say about it later chapters. For reasons that will become clear in Chapter 12, the root of the characteristic equation, s, is also called the *natural frequency* of the system.

We now know that the homogeneous solution is of the form

$$v_{CH} = Ae^{-t/RC} \tag{10.11}$$

The product RC has the dimensions of time and is called the *time constant* of the circuit.

The second step is to find a particular solution, that is, to find any solution v_{CP} that satisfies the original differential equation; it need not satisfy the initial conditions. That is, we are looking for any solution to the equation

$$I_0 = \frac{v_{CP}}{R} + C\frac{dv_{CP}}{dt} \quad . \tag{10.12}$$

Since the drive I_0 is constant in time for $t > 0$, one acceptable particular solution is also a constant:

$$v_{CP} = K \tag{10.13}$$

To verify this, we substitute into Equation 10.12

$$I_0 = \frac{K}{R} + 0 \tag{10.14}$$

$$K = I_0 R \tag{10.15}$$

Because Equation 10.14 can be solved for K, we are assured that our "guess" about the form of the particular solution, that is, Equation 10.13, was correct.[2] Hence the particular solution is

[2] Alternatively, a guess of

$$v_{CP} = Kt$$

where K is a constant independent of t, would not be correct, since substituting into Equation 10.12 yields

$$I_0 = \frac{Kt}{R} + CK$$

which cannot be solved for a time-independent K.

$$v_{CP} = I_0 R \qquad (10.16)$$

The total solution is the sum of the homogeneous solution (Equation 10.11) and the particular solution (Equation 10.16)

$$v_C = Ae^{-t/RC} + I_0 R \qquad (10.17)$$

The only remaining unevaluated constant is A, and we can solve for this by applying the initial condition. Equation 10.5 applies for t less than zero, and our solution, Equation 10.17 is valid for t greater than zero. These two parts of the solution are patched together by a *continuity condition* derived from Equation 9.9: an instantaneous jump in capacitor voltage requires an infinite spike in current, so *for finite current, the capacitor voltage must be continuous.* This circuit cannot support infinite capacitor current (because $i(t)$ is finite, the infinite current would have to come from the resistor, and this is impossible). Thus we are justified in assuming continuity of v_C, hence can equate the solutions for negative time and positive time by solving at $t = 0$

$$0 = A + I_0 R \qquad (10.18)$$

Thus

$$A = -I_0 R \qquad (10.19)$$

and the complete solution for $t > 0$ is

$$v_C = -I_0 Re^{-t/RC} + I_0 R$$

or

$$\boxed{v_C = I_0 R(1 - e^{-t/RC}) \qquad (10.20)}$$

This is plotted in Figure 10.2c.

Some comments at this point help to give perspective. First, notice that capacitor voltage starts from a zero value at $t = 0$ and reaches its final value of $I_0 R$ for large t. The increase from 0 to $I_0 R$ has a time constant RC. The final value of $I_0 R$ for the capacitor voltage implies that all of the current from the current source flows through the resistor, and the capacitor behaves like an *open circuit* (for large t).

Second, the initial value of 0 for the capacitor voltage implies that at $t = 0$ all of the current from the current source must be flowing through the capacitor, and none

through the resistor. Thus the capacitor behaves like an *instantaneous short circuit* at $t = 0$.

Third, the physical significance of the time constant RC can now be seen. Illustrated in Figure 10.4, it is the temporal scale factor that determines how rapidly the transient goes to completion.

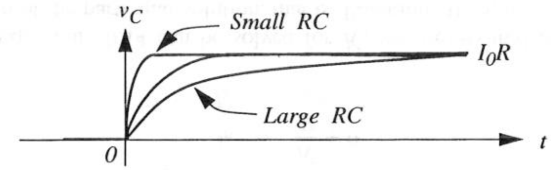


Figure 10.4: Significance of the RC time constant

Finally, it may seem that the solution to such a simple problem can't possibly be as involved as this appears. Correct. This problem and most first order systems with step excitation can be solved by inspection (see Section 10.3). But here we are trying to establish general methods, and have chosen the simplest example to illustrate the method.

10.1.2 RC Discharge Transient

With the capacitor now charged, assume that the current source is suddenly set to zero as suggested in Figure 10.5a, where for convenience, the time axis is redefined so that the turn-off occurs at $t = 0$. The relevant circuit to analyze the RC turn-off or discharge transient now contains just a resistor and a capacitor as indicated in Figure 10.5c. The voltage on the capacitor at the start of the experiment is represented by the initial condition

$$v_C = I_0 R \qquad t < 0 \qquad (10.21)$$

This RC discharge scenario is identical to that of a circuit containing a resistor and a capacitor, where there is an initial voltage $v_C(0) = I_0 R$ on the capacitor.

Because the drive current is zero, the differential equation for t greater than zero is now

$$0 = \frac{v_C}{R} + \frac{C dv_C}{dt} \qquad (10.22)$$

As before, the homogeneous solution is

$$v_{CH} = Ae^{-t/RC} \qquad (10.23)$$

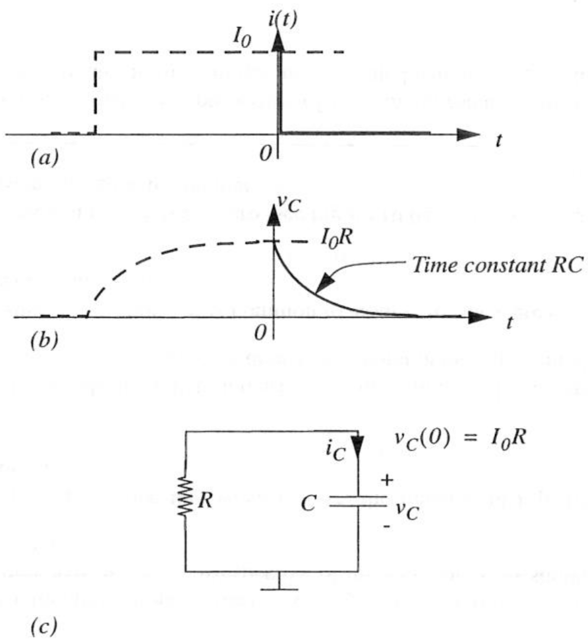


Figure 10.5: RC discharge transient

but now the particular solution is zero, since there is no forcing input, so Equation 10.23 is the total solution. In other words,

$$v_C = v_{CH} = Ae^{-t/RC}$$

Equating Equations 10.21 and 10.23 at $t = 0$, we find

$$I_0R = A \tag{10.24}$$

so the capacitor voltage waveform for $t > 0$ is

$$v_C = I_0Re^{-t/RC} \tag{10.25}$$

This solution is sketched in Figure 10.5b.

In general, for a resistor and capacitor circuit with an initial voltage $v_C(0)$ on the capacitor, the capacitor voltage waveform for $t > 0$ is

$$\boxed{v_C = v_C(0)e^{-t/RC}} \tag{10.26}$$

Properties of Exponentials

Because decaying exponentials occur so frequently in solutions to simple RC and RL transient problems, it is helpful at this point to discuss some of the properties of these functions as an aid to sketching waveforms.

- For a general exponential function of the form

$$x = Ae^{-t/\tau} \tag{10.27}$$

the initial slope of the exponential is

$$\left.\frac{dx}{dt}\right|_{t=0} = \frac{-A}{\tau}$$

Hence the initial slope of the curve, projected to the time axis, intercepts the time axis at $t = \tau$, irrespective of the value of A, as shown in Figure 10.6a.

- Furthermore, notice that when $t = \tau$, the function in Equation 10.27 becomes

$$x(t = \tau) = \frac{A}{e}$$

In other words, the function reaches $1/e$ of its initial value irrespective of the value of A. Figure 10.6b depicts this point in the exponential curve.

- Because $e^{-5} = .0067$, it is common to assume for the t greater than five time constants, i.e.,

$$t > 5\tau$$

the function is essentially zero (see Figure 10.6a). That is, we assume the transient has gone to completion.

We will see later that these properties of the time constant τ make it useful in obtaining rough estimates for time durations associated with rising or falling exponentials.

10.1.3 Series RC Circuit, Step Input

Let us now convert the Norton source in Figure 10.2 to a Thévenin source in Figure 10.7 and determine the capacitor voltage as a function of time. The input waveform v_S is assumed to be a voltage step of magnitude V applied at $t = 0$, but this time around, we assume the capacitor voltage is V_O just before the step.[3] That is, the initial

[3]For the purpose of determining the response for $t \geq 0$, it does not really matter to us how the capacitor voltage became V_O for $t = 0$, or the value of the capacitor voltage for $t < 0$. Nevertheless, below is one possible circuit that will realize the given initial condition on the capacitor and the effect of a step input.

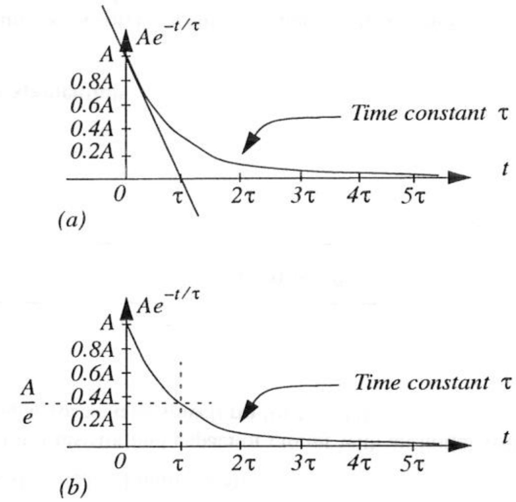


Figure 10.6: Properties of exponentials

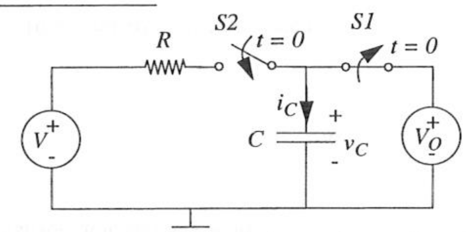
condition on the circuit is

$$v_C = V_O \qquad t < 0 \tag{10.28}$$

The differential equation can be found by using the node method. Applying KCL at the node with voltage v_C, we get

$$\frac{v_C - v_I}{R} + C\frac{dv_C}{dt} = 0$$

Dividing by C and rearranging terms,



In the circuit, a DC source with value V_O is applied across the capacitor using switch S1. The DC source forces the capacitor voltage to V_O. This DC source is switched out as shown at $t = 0$, and another DC source with voltage V is switched in using switch S2. This action applies a step voltage of magnitude V to the capacitor, which has an initial voltage V_O at $t = 0$.

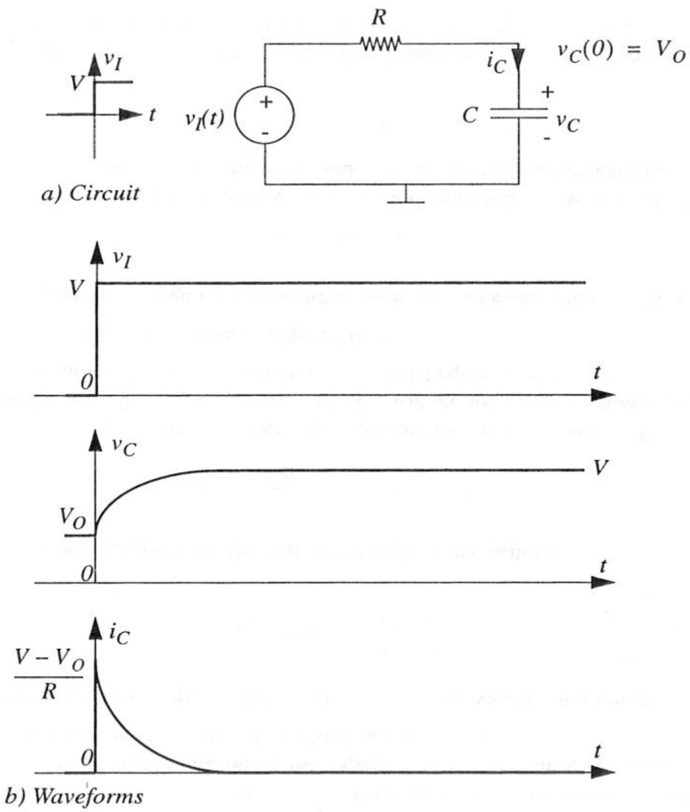


Figure 10.7: Series RC circuit with step input

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{v_I}{RC} \tag{10.29}$$

The homogeneous equation is

$$\frac{dv_{CH}}{dt} + \frac{v_{CH}}{RC} = 0 \tag{10.30}$$

which, as expected, is the same as that in Equation 10.6 for the Norton circuit, since the Norton and Thévenin circuits are equivalent. Borrowing the homogeneous solution to Equation 10.6, we have

$$v_{CH} = Ae^{-t/RC} \tag{10.31}$$

where RC is the time constant of the circuit.

Let us now find the particular solution. Since the input drive is a step of magnitude V, the particular solution is any solution to

$$\frac{dv_{CP}}{dt} + \frac{v_{CP}}{RC} = \frac{V}{RC} \tag{10.32}$$

Because the drive is a step, which is constant for large t, we can assume a particular solution of the form

$$v_{CP} = K \tag{10.33}$$

Substituting into Equation 10.32, we obtain

$$\frac{K}{RC} = \frac{V}{RC}$$

which implies $K = V$. So the particular solution is

$$v_{CP} = V \tag{10.34}$$

Summing v_{CH} and v_{CP}, we obtain the complete solution

$$v_C = V + Ae^{-t/RC} \tag{10.35}$$

The initial condition can now be applied to evaluate A. Given that the capacitor voltage must be continuous at $t = 0$, we have

$$v_C(t = 0) = V_O$$

Thus, at $t = 0$, Equation 10.35 yields

$$A = V_O - V$$

The complete solution for the capacitor voltage for $t > 0$ is now

$$v_C = V + (V_O - V)e^{-t/RC} \tag{10.36}$$

where, V is the input drive voltage for $t > 0$ and V_O is the initial voltage on the capacitor. As a quick sanity check, substituting $t = 0$, we get $v_C(0) = V_O$, and substituting $t = \infty$, we get $v_C(\infty) = V$. Both these boundary values are what we expect, since the initial condition on the capacitor is V_O, and since the input voltage must appear across the capacitor after a long period of time.

By rearranging the terms, Equation 10.36 can be equivalently written as

$$v_C = V_O e^{-t/RC} + V(1 - e^{-t/RC}) \tag{10.37}$$

Finally, from Equation 9.9, the current through the capacitor is

$$i_C = C\frac{dv_C}{dt} = \frac{V - V_O}{R}e^{-t/RC} \tag{10.38}$$

This expression for i_C also matches our expectations since i_C must be 0 when t is large, and the since the capacitor behaves like a voltage source with voltage V_O during the step transition at $t = 0$, the current at $t = 0$ must equal $(V - V_O)/R$.

These waveforms are shown in Figure 10.7b.

If we desire the voltage v_R across the resistor, we can easily obtain it by applying KVL as

$$v_R = v_I - v_C$$

where we take the positive reference for v_R on the input side of the resistor. Alternatively, we can obtain v_R by taking the product of the current and the resistance as

$$v_R = i_C R$$

As one final point of interest, notice that Equation 10.36 was derived assuming both an initial nonzero state (V_O) and a nonzero input (a step of voltage V).

Substituting $V = 0$ in Equation 10.36 we obtain the so called *zero input response* (ZIR):

$$v_C = V_O e^{-t/RC} \tag{10.39}$$

and substituting $V_O = 0$ in Equation 10.36 we obtain the *zero state response* (ZSR):

$$v_C = V - Ve^{-t/RC} \tag{10.40}$$

In other words, the zero input response is the response for nonzero initial conditions, but where the input drive is zero. In contrast, the zero state response is the response of the circuit when the initial state is zero, that is, all capacitor voltages and inductor currents are initially zero.

Notice also that the total response is the sum of the ZIR and the ZSR,

as can be verified by adding the right hand sides of Equations 10.39 and 10.40 and comparing to the right hand side of Equation 10.36. We will have a lot more to say about the ZIR and the ZSR in Section 10.5.3.

10.1.4 Series RC Circuit, Square Wave Input

Examination of the waveforms in Figure 10.5a and 10.5b indicates that the presence of the capacitor has *changed the shape* of the input wave. When a square pulse is applied to the RC circuit, a decidedly non-square pulse, with slow rise and slow decay, results. The capacitor has allowed us to do a limited amount of wave shaping. This concept can be further developed by an experiment in which we drive the circuit with a square wave.

In this experiment, we will use a Thévenin source as in Figure 10.8. The source can be a standard laboratory square-wave generator. The input square wave is marked as a in Figure 10.8. Several quite distinctive wave shapes for $v_C(t)$ can be derived, depending on the relation between the *period* of the driving square wave and the *time constant* RC of the network. These waveforms are all essentially variations on the solution derived in the preceding sections.

For the case where the circuit time constant is very short compared to the square wave period, the exponentials go to completion relatively rapidly, as suggested by waveform b in Figure 10.8. The capacitor waveform thus closely resembles the input waveform, except for a small amount of rounding at the corners.

If the time constant is a substantial fraction of the pulse length, then the solution appears as waveform c in Figure 10.8. Note that the drawing implies that the transients still go almost to completion, so there is an upper limit on the RC product for this solution to apply. Assuming, as noted above, that simple transients are complete for times greater than five time constants, the RC product must be less than one fifth of the pulse length, or one tenth the square-wave period for this solution to apply.

When the circuit time constant is much longer than the square-wave period, waveform d shown in Figure 10.8 results. Here the transient clearly does *not* go to completion. In fact, only the first part of the exponential is ever seen. The waveform looks almost triangular, the *integral* of the input wave. This can be seen from the differential equation describing the circuit. Application of KVL gives

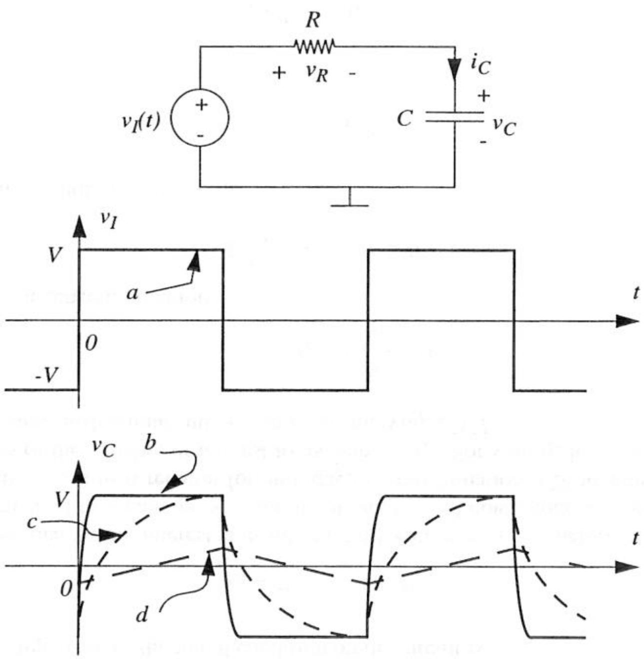


Figure 10.8: Response to square wave

$$v_I = i_C R + v_C \tag{10.41}$$

Upon substitution of the constituent relation for the capacitor, Equation 9.9, we obtain the differential equation

$$v_I = RC\frac{dv_C}{dt} + v_C \tag{10.42}$$

It is clear from Equation 10.42 or Figure 10.8 that as the circuit time constant becomes bigger, the capacitor voltage v_C must become smaller. For waveform d the time constant RC is large enough that v_C is much smaller than v_I, so in this case Equation 10.41 can be approximated by

$$v_I \simeq i_C R \tag{10.43}$$

Physically, the current is now determined solely by the drive voltage and the resistor, because the capacitor voltage is almost zero. Integrating both sides of Equation 10.42 assuming v_C is negligible, we obtain

$$v_C \simeq \frac{1}{RC}\int v_I dt + K \tag{10.44}$$

where the constant of integration K is zero. Thus for large RC, the capacitor voltage is approximately the integral of the input voltage. This is a very useful signal-processing property. In Chapter 15 we will show that a much closer approximation to ideal integration can be obtained by adding an Op Amp to the circuit.

It is a simple matter to find the voltage across the resistor in the circuit of Figure 10.8 because we can find the current from the capacitor voltage using Equation 9.9,

$$v_R = i_C R = RC\frac{dv_C}{dt}$$

Thus, during the charge interval, for example, from Equation 10.20, assuming the transients go to completion,

$$v_C = V(1 - e^{-t/RC})$$

Hence

$$v_R = Ve^{-t/RC}$$

The wave shapes in Figure 10.8 change very little if the input signal v_I has zero average value, that is, if v_I is changed so that it jumps back and forth from $-V/2$ to $+V/2$. Specifically, v_C also has zero average value, and if the transients go to completion, as in b and c, the excursions will be $-V/2$ and $+V/2$.

10.2 Analysis of RL Circuits

10.2.1 Series RL Circuit, Step Input

Figure 10.9 will serve as a simple illustration of a transient involving an inductor. (See the example discussed in Section 10.6.1 for a practical application of the analysis involving inductor transients.) The input waveform v_S is assumed to be a voltage step applied at $t = 0$ (see Figure 10.9a), and the inductor current is assumed to be zero just before the step. That is, the initial condition on the circuit is

$$i_L = 0 \qquad t < 0 \tag{10.45}$$

Suppose that we are interested in solving for the current i_L. As before, we can use the node method to obtain an equation involving the unknown node voltage v_L, and then use the constituent relation for an inductor from Equation 9.28 to substitute for v_L in terms of the variable of interest to us, namely i_L. For variety, however, we will derive the same differential equation in i_L by applying KVL:

$$-v_S + i_L R + L\frac{di_L}{dt} = 0 \tag{10.46}$$

The homogeneous equation is

$$L\frac{di_{LH}}{dt} + i_{LH}R = 0 \tag{10.47}$$

Assume a solution of the form

$$i_{LH} = Ae^{st} \tag{10.48}$$

Hence

$$LsAe^{st} + RAe^{st} = 0 \tag{10.49}$$

For non-zero A ($A = 0$ is a trivial solution)

$$Ls + R = 0$$

or

$$s + \frac{R}{L} = 0 \tag{10.50}$$
$$s = -R/L \tag{10.51}$$

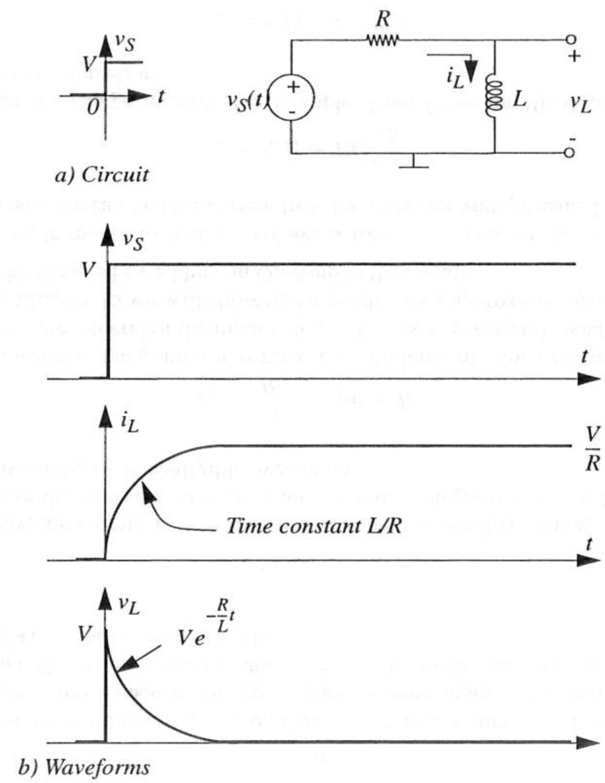


Figure 10.9: Inductor current buildup

Chapter 13

Sinusoidal Steady State: Impedance and Frequency Response

13.1 Introduction

This chapter represents a major change in point of view for circuit analysis, hence it is important to review where we have been and where we are going. The analysis method discussed in preceding chapters has four basic steps:

- Draw a circuit model of the problem
- Formulate the differential equations
- Solve these equations. If the equations are linear, then find the homogeneous solution and the particular solution. If the equations are nonlinear, then numerical methods often are required.
- Use the initial conditions to evaluate the constants in the homogeneous solution.

This approach, diagrammed in the top of Figure 13.1, is basic and powerful, in that it can handle both linear and nonlinear problems, but often it involves substantial mathematical manipulations if the drive signals are other than simple impulses, steps or ramps. Thus there is considerable incentive to look for easier methods of solution, even if these methods are more restricted in application. Simplified methods are indeed possible if the system is linear and time invariant, and we assume *sinusoidal* drive and focus on the steady-state behavior. Because in many design applications such as audio amplifiers, oscilloscope vertical amplifiers, Op Amps, etc., linearity is a basic design constraint, systems which are linear or at least incrementally linear represent a large and important class, hence are worthy of special attention. Further, more complicated

input signals such as square waves can be considered to be the sum of many sinusoids, hence the problem can be solved by superposition.

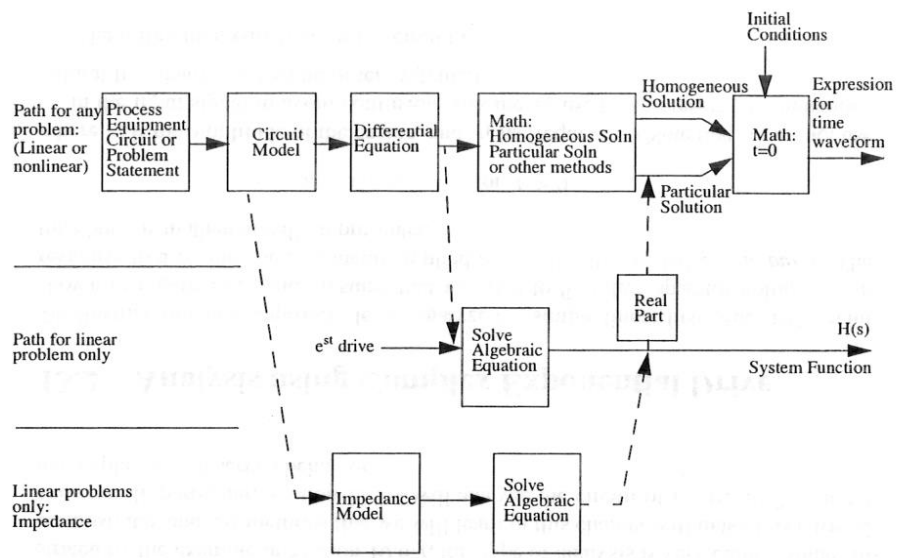


Figure 13.1: Analysis methods

Equally important, we often characterize systems by their frequency response (i.e., sinusoidal response). Examples include our hearing, audio equipment, ultrasonic pest deterrents, wireless network receivers etc. The frequency related behavior of such systems is as important as their time domain behavior. Therefore, the sinusoidal steady state response is useful because it is a natural and convenient way to describe the behavior of linear systems.

We wish to show in this chapter that the solution to linear circuit problems is greatly simplified by assuming a drive of the form e^{st} as illustrated in the center panel of Figure 13.1, primarily because under this assumption the differential equation is transformed into an algebraic equation, and because the response to a sinusoidal drive can be directly obtained from the response to the e^{st} drive. This leads further to a shorthand solution method involving the concept of *impedance*, whereby the algebraic equation can be found directly from the circuit model, without writing the differential equation at all, as diagrammed in the lowermost panel of the figure.

The insight behind the employment of a drive of the form e^{st}, where $s = j\omega$, is this. Recall, we wish to find the system response in the steady state[1] to a sinusoidal input of the form $\cos \omega t$. We will show that directly solving system differential equations with a sinusoidal input leads to a tangle of trigonometry and is very complicated. (You have already seen an example of a direct solution of an RL circuit for a sinusoidal drive in Section 10.6.7.) Instead, we employ the following mathematical trick: realizing that

$$e^{j\omega t} = \cos \omega t + j \sin \omega t \tag{13.1}$$

(the Euler relation), we first obtain with relative ease the circuit response to an unrealizable drive of the form $e^{j\omega t}$. The resulting response will contain a real part and an imaginary part. For real linear systems, by superposition, the real part of the response is due to the real part of the input (namely, $\cos \omega t$) and the imaginary part of the response is due to the imaginary part of the input (namely, $j \sin \omega t$). Accordingly, by taking the real part of the response to $e^{j\omega t}$, we obtain the response to a real sinusoidal input of the form $\cos \omega t$. (Similarly, by taking the imaginary part of the response to $e^{j\omega t}$, we obtain the response to an input of the form $\sin \omega t$.)

To motivate the study of methods based on the sinusoidal steady state, let us present an example of the type of problem that can be solved with ease using these methods. Suppose we construct the linear small signal amplifier shown in Figure 13.2 by concatenating two single stage MOSFET amplifiers of the type studied in Chapter 8. The DC voltage V_I is chosen to bias the first stage appropriately, and the DC value of the first stage output voltage V_O provides the bias for the second stage. The figure further shows the presence of a capacitor C_{GS} at the input node of the second stage (for example, reflecting the gate capacitance of the MOSFET in the second stage).

Suppose, now, that we wish to find the first-stage output voltage v_o in response to v_i, a small sinusoidal signal applied to the input of the amplifier. In particular, we are interested in determining how the presence of the capacitor C_{GS} affects the amplification afforded by the first amplifier stage. Suppose, further, that we do not care about initial transients, rather, we are interested in the steady-state behavior when all transients have died out. Experimental application of a sinusoid to the input and measurement of the response v_o will show very different behavior as the frequency of the input is swept from a low to a high value. We will observe that for low frequency signals the gain of the first stage is no different from our earlier calculations in Chapter 8 in the absence of the capacitor C_{GS}. However, we will also observe that the presence of the capacitor makes the gain of the amplifier fall off rapidly at high frequencies.

[1] Interestingly, the substitution of $s = j\omega$ will give us the response of the circuit in sinusoidal steady state. Although not covered in this book, the use of Laplace Transforms where we substitute $s = \sigma + j\omega$ will yield the total response.

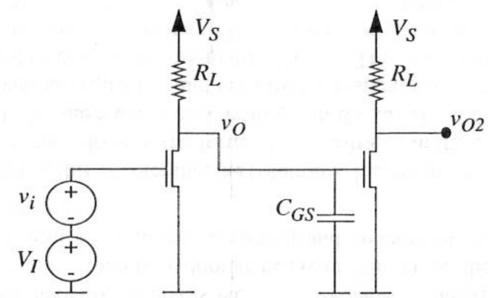


Figure 13.2: A two stage MOSFET amplifier showing the MOSFET gate capacitor

Analytical analysis based on the methods we learned in the previous chapters would suggest writing the differential equation for the circuit comprising the resistor R_L and the capacitor C_{GS} and finding the forced response to an applied sinusoid. As demonstrated by the example in Section 10.6.7, this type of analysis is very cumbersome. In contrast, the analysis methods that we will learn in this chapter will make this a trivial exercise. In particular, Section 13.3.4 will analyze the circuit of Figure 13.2 in detail and explain the observed behavior.

13.2 Analysis using Complex Exponential Drive

To illustrate this new approach, let us analyze the simple linear first-order RC circuit shown in Figure 13.3, and presume that we wish to find the capacitor voltage v_c, in response to a cosine wave suddenly applied at $t = 0$, often called a *tone burst*. The tone burst is mathematically represented as

$$v_i = V_i \cos \omega_1 t \quad \text{for } t \geq 0 \quad ,$$

where V_i is the amplitude of the cosine, and ω_1 its frequency. (Note that we do not use ω_o in the input signal to avoid confusion with the ω_o used to represent the undamped natural frequency in a second order systems.)

The differential equation for the circuit is

$$v_i = v_c + RC\frac{dv_c}{dt} \tag{13.2}$$

Let us attempt to solve this differential equation by summing its homogeneous and particular solutions. Recall, when dealing with circuit responses, the homogeneous solution is also called the natural response, and the particular solution is also called

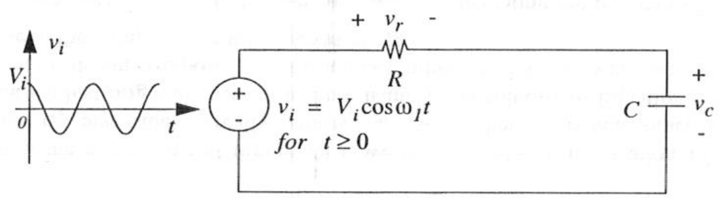


Figure 13.3: RC circuit with tone burst in. The amplitude of the input waveform is V_i, where V_i is real

the forced response. Recall further that the forced response depends on the external inputs to the circuit. Let us denote the homogeneous solution as v_{ch} and the particular or forced solution as v_{cp}. Then, we know that the total solution is given by

$$v_c = v_{ch} + v_{cp}$$

13.2.1 Homogeneous Solution

From Equation 13.2, the homogeneous solution can be derived by solving

$$RC\frac{dv_{ch}}{dt} + v_{ch} = 0 \tag{13.3}$$

As we have seen in Chapter 10, the homogeneous solution for this equation is

$$v_{ch} = K_1 e^{-t/RC} \tag{13.4}$$

where K_1 is a constant to be determined from the initial conditions.

13.2.2 Particular Solution

The straightforward approach to finding the particular or forced solution v_{cp} involves finding any solution to the differential equation

$$v_i = v_{cp} + RC\frac{dv_{cp}}{dt} \tag{13.5}$$

Since the input v_i is given by

$$v_i = V_i \cos\omega_1 t$$

(where V_i is real), this amounts to finding any solution to

$$V_i \cos\omega_1 t = v_{cp} + RC\frac{dv_{cp}}{dt} \tag{13.6}$$

Obviously the forced response v_{cp} must be some combination of sines and cosines, so we assume

$$v_{cp} = K_2 \sin\omega t + K_3 \cos\omega t \tag{13.7}$$

or, equivalently,

$$v_{cp} = K_4 \cos(\omega t + \Phi) \tag{13.8}$$

There is nothing wrong with this approach, except that it leads to a tangle of trigonometry. So, we will abandon this path.

Instead, let us launch out in a slightly different direction. The Euler Relation

$$e^{j\omega t} = \cos\omega t + j\sin\omega t \tag{13.9}$$

shows that $e^{j\omega t}$ contains the cosine term we want, in addition to an unwanted sine term. Hence, by a sort of inverted superposition argument, we replace the actual source v_i with a source of the form

$$\tilde{v_i} = V_i e^{s_1 t} \tag{13.10}$$

and return later to unscramble the cosine and sine parts. In this equation we have used

s_1 as a shorthand for $j\omega_1$,

and have included a "~" above v_i to indicate that this is not the true drive voltage. For consistency, we will use the same notation for all variables related to this fake drive voltage. The differential equation to find the particular solution to $\tilde{v_i}$ now becomes

$$\tilde{v_i} = V_i e^{s_1 t} = \tilde{v_{cp}} + RC\frac{d\tilde{v_{cp}}}{dt} \tag{13.11}$$

It is clear that a reasonable assumption for the particular solution is

$$\tilde{v_{cp}} = V_c e^{st} \tag{13.12}$$

in which we must somehow find V_c and s. On substitution of the assumed particular solution into Equation 13.11, we obtain

$$V_i e^{s_1 t} = V_c e^{st} + RCsV_c e^{st} \tag{13.13}$$

We note first that s must equal s_1, otherwise Equation 13.13 cannot be satisfied for all time. Now, on the basis that e^{st} can never be zero for finite values of t, the $e^{s_1 t}$ terms can be divided out, to yield an *algebraic equation relating the complex amplitudes of the voltages* rather than a differential equation relating the voltages as time functions:

$$V_i = V_c + V_c RC s_1 \tag{13.14}$$

which can be solved to yield

$$\boxed{V_c = \frac{V_i}{1 + RC s_1} \quad \text{for} \quad s_1 \neq -\frac{1}{RC}} \tag{13.15}$$

a restriction clearly satisfied in this case because $s_1 = j\omega_1$ where ω_1 is a real number. Thus Equation 13.15 becomes

$$V_c = \frac{V_i}{1 + j\omega_1 RC} \tag{13.16}$$

or, from Equation 13.12, the particular solution for the fake input $\tilde{v}_i$ is

$$\tilde{v_{cp}} = \frac{V_i}{1 + j\omega_1 RC} e^{j\omega_1 t} \tag{13.17}$$

At this point the reader should protest. No waveform measured in the laboratory will have a "j" associated with it. The problem arises because we have used a complex rather than a real drive. That is, we have analyzed the circuit shown in Figure 13.4a, rather than Figure 13.3. The complex exponential drive $\tilde{v}_i$ can be represented by the Euler Relation as the sum of two sources as depicted in Figure 13.4a. *If the circuit is linear*, the two-source circuit can be analyzed by superposition, as suggested in Figure 13.4b and c. Specifically, the voltage $\tilde{v_{cp}}$ can be found by summing the response to $V_i \cos \omega t$, as obtained from b with j times the response to $V_i \sin \omega t$, as found from c

$$\tilde{v_{cp}} = v_{cp1} + j v_{cp2} \tag{13.18}$$

From the perspective of Figure 13.3, we have calculated in Equation 13.18 the response $\tilde{v_{cp}}$, and what we really want is v_{cp1}. Notice that v_{cp1} is none other than the v_{cp} that we had originally set out to find, namely, the solution to Equation 13.6. So we want to "de-superimpose" the two sources in Figure 13.4a. This is a simple matter because of the j flag: v_{cp1} is the real part of $\tilde{v_{cp}}$.

The next task, then, is to find an easy way of calculating the real part of a complex expression. (Those readers who are a little hazy about manipulation of complex

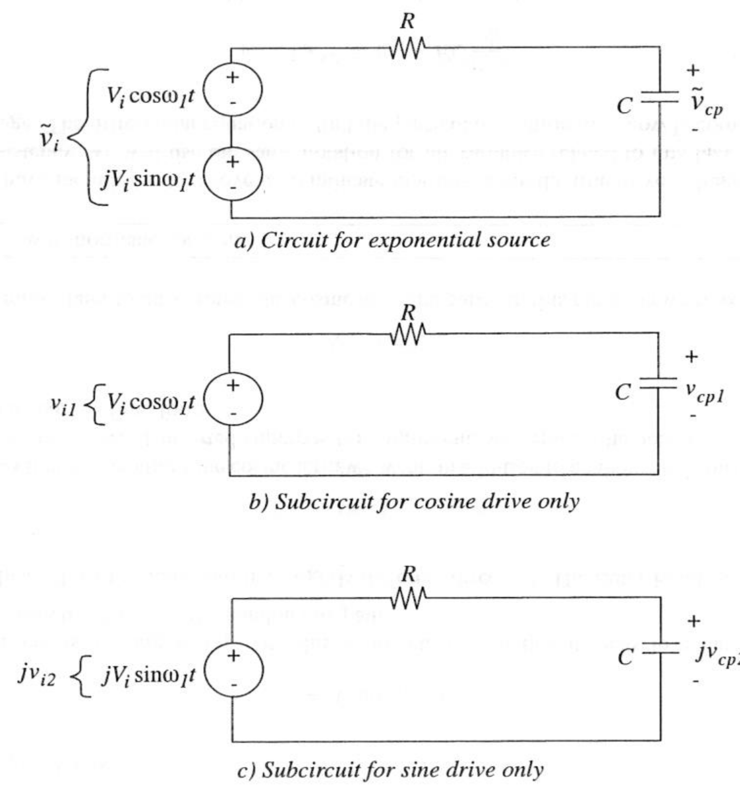


Figure 13.4: RC circuit with exponential drive e^{st}

numbers, and in particular the conversion between rectangular and polar form, should review at this point Appendix C on complex numbers or a suitable math text.) In this specific problem, we must find the real part of the $\tilde{v_{cp}}$ expression, Equation 13.17. The difficulty is that the expression has two factors, one in Cartesian or rectangular form, and the other in polar, whereas multiplication is simpler if both factors are in polar form. Hence we rewrite Equation 13.17 in polar form as

$$\tilde{v_{cp}} = \frac{V_i}{\sqrt{1 + (\omega_1 RC)^2}} e^{j\Phi} e^{j\omega_1 t} \tag{13.19}$$

where

$$\Phi = \tan^{-1} \frac{-\omega RC}{1} \tag{13.20}$$

Now, to find the real part of $\tilde{v_{cp}}$, we use the Euler relation to write Equation 13.19 as

$$\tilde{v_{cp}} = \frac{V_i}{\sqrt{1 + (\omega_1 RC)^2}} \cos(\omega_1 t + \Phi) + j \frac{V_i}{\sqrt{1 + (\omega_1 RC)^2}} \sin(\omega_1 t + \Phi)$$

from which the real part of $\tilde{v_{cp}}$ is available by inspection,

$$v_{cp1} = \frac{V_i}{\sqrt{1 + (\omega_1 RC)^2}} \cos(\omega_1 t + \Phi) \tag{13.21}$$

This, finally, is the particular solution of Equation 13.6.

13.2.3 Complete Solution

The complete expression for the capacitor voltage in response to a cosine tone burst is the sum of this particular solution (v_{cp1}) and the homogeneous solution (v_{ch}) previously found in Equation 13.4:

$$v_c = K_1 e^{-t/RC} + \frac{V_i}{\sqrt{1 + (\omega_1 RC)^2}} \cos(\omega_1 t + \Phi) \tag{13.22}$$

The one remaining unknown constant, K_1, can be found from the initial conditions by setting t to zero in the usual manner. However, as we will see shortly, we usually do not care about the first term.

13.2.4 Sinusoidal Steady State Response

Under sinusoidal drive, we are almost always interested in the *steady-state* value of the capacitor voltage, which can be readily obtained from Equation 13.22 by assuming t is very large. When $t \to \infty$, Equation 13.22 reduces to

$$v_c = \frac{V_i}{\sqrt{1 + (\omega_1 RC)^2}} \cos(\omega_1 t + \Phi) \tag{13.23}$$

which is simply the particular solution to a cosine input (compare with Equation 13.21). For a cosine input, the steady-state response is often termed the response to a cosine. The corresponding complete response is termed the response to a cosine burst, and includes both the homogeneous and particular terms. In Equation 13.23, the $V_i/\sqrt{1 + (\omega_1 RC)^2}$ factor gives the amplitude (or magnitude) of the response, and Φ is the phase. The phase is the angular difference between the output and input sinusoids. Notice that both the magnitude and phase (see Equation 13.20) of the response are frequency dependent.

Equation 13.22 is really quite general, in that it gives the capacitor voltage for any amplitude and any frequency of cosine tone burst. For example, it is obvious that at low frequencies, (that is, for ω_1 small), and after the transient has died away

$$v_c \simeq V_i \cos \omega_1 t \tag{13.24}$$

Thus, after the transient has died away, the output looks almost like the input. We conclude that for ω_1 small, the capacitor behaves like an open circuit. Further, for ω_1 large, i.e., at high frequencies, after the transient has died away

$$v_c \simeq \frac{V_i}{\omega_1 RC} \cos(\omega_1 t - 90^\circ) \tag{13.25}$$

so the output will be sinusoidal, but 90 degrees out of phase with the input, and much smaller. At high frequencies then, the magnitude of the capacitor voltage will get very small, so we can say that the capacitor begins to behave like a short circuit.

There are four general conclusions to be drawn from this specific example.

1. The use of an e^{st} drive reduces a differential equation to an algebraic equation, thereby simplifying the solution. This solution process replaces trigonometry with complex algebra, which is a wise trade.

2. The last couple of pages from Equation 13.17 to Equation 13.22, although necessary for completeness, did not add any new insight about the circuit behavior. For example, the same information about the form of v_c in the steady state, or

its value at low frequencies and high frequencies could have been found from Equation 13.17, or even from the *complex amplitude* V_c, Equation 13.16, just as easily as from Equation 13.21, without the intervening "real part" calculation.[2]

For example, the steady state value of v_c (or the particular or forced response) can be determined from the value of V_c as

$$v_c = Re\left[V_c e^{j\omega_1 t}\right] \qquad (13.26)$$

or, equivalently,

$$v_c = |V_c| \cos(\omega_1 t + \angle V_c) \qquad (13.27)$$

Figure 13.5 shows a sketch of the input cosine and the output response with the various magnitudes and phases marked. Notice that the complex amplitude V_c carries both the amplitude and phase information of the response ($|V_c|$ and $\angle V_c$ respectively) in an easily accessible manner. Thus, our analysis can stop at Equation 13.16.

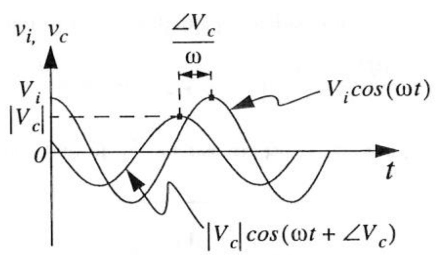


Figure 13.5: The amplitude and phase of the response v_c compared to the input sinusoid v_i

3. The denominator of the V_c expression, Equation 13.15, has the same form as the characteristic polynomial in the homogeneous solution, (see Chapter 10, Equation 10.9 for example) so the value of s in the homogeneous solution could have been found from this denominator without any formal solution of the homogeneous equation. That this is a general result can be shown by examining the two derivations.

2 Recall from Equation 13.16 that V_c is the complex amplitude of the forced response to our fake input $\tilde{v}_i = V_i e^{j\omega_1 t}$.

4. The V_c expression, Equation 13.15, looks very much like a voltage divider expression, especially if we divide through by Cs_1.

$$V_c = \frac{1/Cs_1}{R + 1/Cs_1} V_i \qquad (13.28)$$

This suggests a very simple method for finding the complex amplitude V_c directly from the circuit: Redraw the circuit, replacing resistors with R boxes, capacitors with $1/Cs_1$ boxes, and cosine sources by their amplitudes, in this case V_i, as shown in Figure 13.6. Now V_c can be found in one line. But what are these boxes? And what is V_c? The next section will provide the answers.

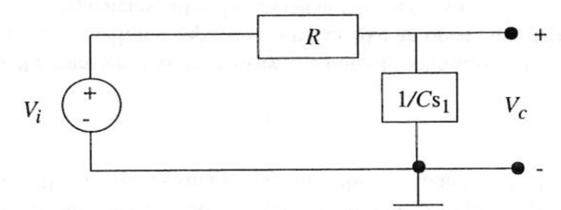


Figure 13.6: A circuit interpretation of Equation 13.15

13.3 The Boxes: Impedance

To get a better idea of the meaning of the boxes in Figure 13.6, let us examine some trivial cases, as sketched in Figure 13.7. In (a), a voltage source $V_i \cos\omega_1 t$ is connected across a capacitor, hence

$$i = C\frac{dv}{dt} \qquad (13.29)$$

On the basis of Section 13.1, assume the voltage and current are of the form

$$v = Ve^{st} \qquad (13.30)$$
$$i = Ie^{st} \qquad (13.31)$$

where, as before, we use s as a shorthand notation for $j\omega$.

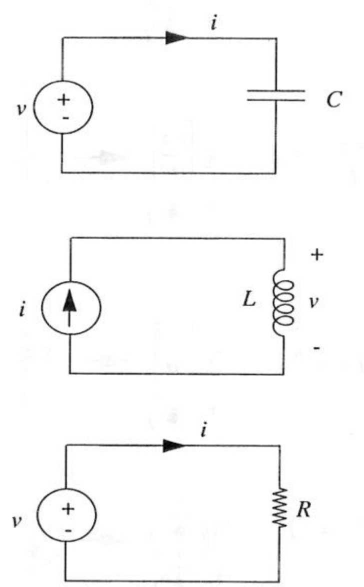


Figure 13.7: Impedance calculations

On substituting these relations in Equation 13.29, and dividing by e^{st} (never zero for finite s and t) we find

$$I = CsV \tag{13.32}$$

or

$$V = \frac{1}{Cs} I \tag{13.33}$$

Similar calculations on the inductor and the resistor yield

$$V = LsI \tag{13.34}$$
$$V = IR \tag{13.35}$$

These equations indicate that for linear R, L or C, in each case the complex amplitude of the voltage is related to the complex amplitude of the current by very simple algebraic expressions which are generalizations of Ohm's Law. The constants relating V to I in Equations 13.33, 13.34, and 13.35 are called *impedances*, and these equations are the constituent relations for C, L, and R expressed in impedance form. The constituent relations for these elements and for voltage and current sources are summarized in Figure 13.8.

Just as we used R to denote resistances, we commonly use the letter Z to denote impedances.

Thus, the impedances of an inductor, a capacitor and a resistor are given by

$$Z_L = sL = j\omega L \tag{13.36}$$

$$Z_C = \frac{1}{sC} = \frac{1}{j\omega C} \tag{13.37}$$

and

$$Z_R = R \tag{13.38}$$

respectively.

Furthermore, just as the conductance was defined as the reciprocal of resistance, we define admittance as the reciprocal of impedance.

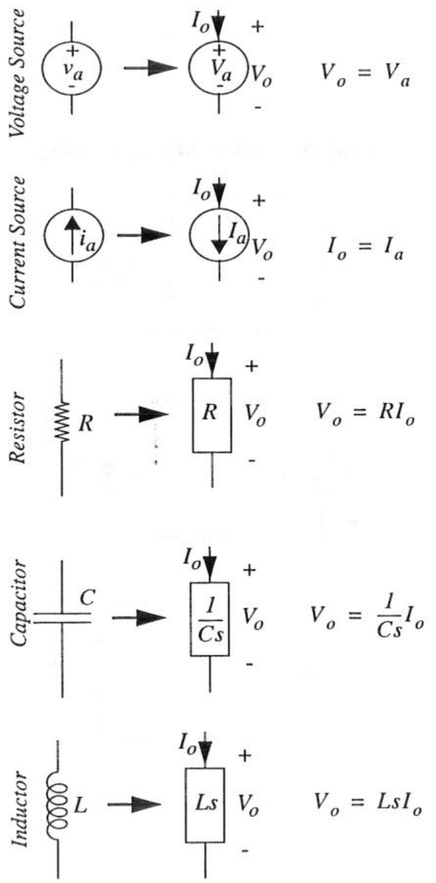


Figure 13.8: Constituent relations for a voltage source, a current source, and R, L and C in impedance form. Note that V_o and I_o are terminal variables, while V_a and I_a are element parameters. Note also that $s = j\omega$

Impedances are complex numbers in general. They are also frequency dependent. Figure 13.9 plots the magnitude of the impedances of an inductor, a capacitor and a resistor as a function of frequency. The curves in the figure reinforce the following intuition developed in Chapter 10 and summarized in Section 10.8.

Inductors behave like short circuits for DC (or very low frequencies) and like open circuits for very high frequencies. Capacitors behave like open circuits for DC (or very low frequencies) and like short circuits for very high frequencies.

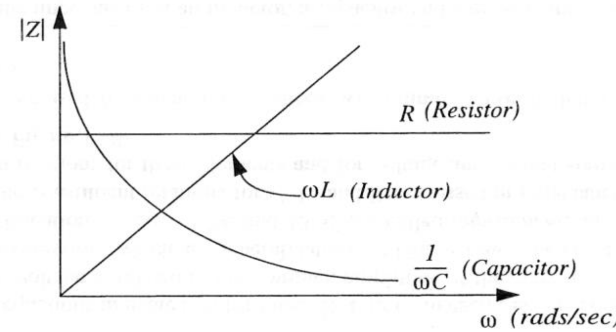


Figure 13.9: Frequency dependence of the impedances of inductors, capacitors, and resistors

Now, generalizing from these results, the relations among complex amplitudes of voltages and currents for any linear RLC network can be found by replacing the (sinusoidal) sources by their complex (or real) amplitudes, and replacing resistors by R boxes, capacitors by $1/Cs$ boxes, and inductors by Ls boxes. The resultant diagram is called the *impedance model* of the circuit. The complex voltages and currents in circuits can now be found by standard linear circuit analysis: Node Equations, Thévenin's theorem, etc.

The impedances follow the same combination rules as resistors, for example, impedances in series add, although here the addition involves complex numbers.

Therefore, the intuitive method based on series and parallel simplifications also applies.

We note that the impedance representation does not change the topology of the circuit – devices are simply replaced by their corresponding impedance models drawn as boxes. The reason is that KVL and KCL apply to a given circuit irrespective of the form of the drive. In other words, KVL and KCL apply irrespective of whether the voltages and currents are sinusoids, DC values, or any other form for that matter. The impedance form simply assumes sinusoidal drive and response and captures in a convenient form individual device behavior when the drives are sinusoids. Thus, because the KVL and KCL equations are unchanged for sinusoidal drive, the circuit topology remains the same because it captures the same information as expressed by KVL and KCL.

If desired, the expressions for the actual voltages and currents, the particular solutions or forced responses in Chapter 10 parlance, can be found by multiplying the corresponding complex variable by $e^{j\omega t}$ and taking the real part.

For example, to obtain the actual voltage $v_x(t)$ from the corresponding complex variable $V_x(j\omega)$, we use

$$v_x(t) = Re\left[V_x(j\omega)\, e^{j\omega t}\right] \tag{13.39}$$

or equivalently,

$$v_x(t) = |V_x| \cos(\omega t + \angle V_x) \tag{13.40}$$

We emphasize again, however, that this step is usually not necessary, because the complex amplitude expression contains all the key information about circuit behavior.

At this point it is necessary to explicitly introduce a notation for voltages and currents to clearly differentiate complex amplitudes from time functions. We abide by the international standard in this matter:

- DC or operating-point variables: upper-case symbols with upper-case subscripts (e.g., V_A)
- Total instantaneous variables: lower-case symbols with upper-case subscripts (e.g., v_A)
- Incremental instantaneous variables: lower-case symbols with lower-case subscripts (e.g., v_a)
- Complex amplitudes or complex amplitudes of incremental components, and real amplitudes of sinusoidal input sources: upper-case symbols with lower-case subscripts (e.g., V_a)

Summarizing, the impedance method allows us to determine with ease the steady-state response of any linear RLC network for a sinusoidal input. The method works with complex amplitudes of voltages and currents at its variables and has the following general steps.

The Impedance Method:

1. First, replace the (sinusoidal) sources by their complex (or real) amplitudes. For example, the input voltage $v_A = V_a \cos(\omega t)$ is replaced by its real amplitude V_a.
2. Replace circuit elements by their impedances, namely, resistors by R boxes, inductors by Ls boxes and capacitors by $1/Cs$ boxes. Here $s = j\omega$. The resulting diagram is called the impedance model of the network.
3. Now, determine the complex amplitudes of the voltages and currents in the circuit by any standard linear circuit analysis technique - Node method, Thévenin method, intuitive method based on series and parallel simplifications, etc.
4. Although this step is not usually not necessary, we can then obtain the time variables from the complex amplitudes. For example, the time variable corresponding to node variable V_o is given by

$$v_O(t) = |V_o| \cos(\omega t + \angle V_o) \tag{13.41}$$

Example 13.1 Revisiting the RC Example

To illustrate the power of the method, let us revisit the RC circuit from Figure 13.3 (redrawn here as Figure 13.10a for convenience) and analyze it using the impedance method just described. As before, suppose that we wish to find the steady-state capacitor voltage v_c in response to an input of the form $v_i = V_i \cos \omega_1 t$.

Figure 13.10b shows the corresponding impedance model. In the model, notice that we have replaced the input voltage v_i with the real amplitude V_i, and the capacitor voltage v_c with the complex amplitude V_c, according to the first step of the impedance method. Further, according to the second step of the method, we have replaced the resistor with an R box and the capacitor with a box with impedance $1/Cs$. As before, s is a shorthand notation for $j\omega$.

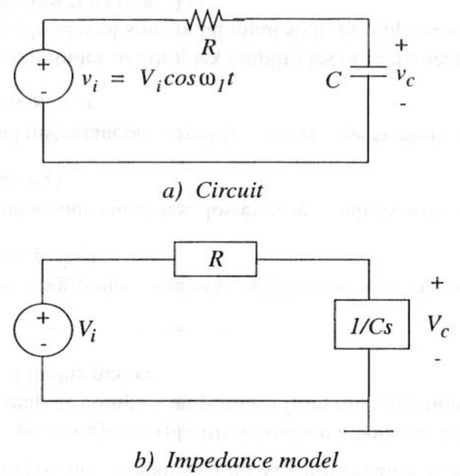


Figure 13.10: Impedance model of RC circuit with sinusoidal input

We can derive an expression for V_c by applying the generalized voltage divider relation in the impedance model in Figure 13.10b as

$$V_c = \frac{Z_C}{Z_R + Z_C} V_i \tag{13.42}$$

where Z_R and Z_C are the impedances of the resistor and the capacitor respectively. Substituting the actual impedance values, we obtain

$$V_c = \frac{1/Cs}{R + 1/Cs} V_i = \frac{1}{RCs + 1} V_i \tag{13.43}$$

Since s is a shorthand for $j\omega$, at a specific frequency ω_1,

$$V_c = \frac{1}{1 + j\omega_1 RC} V_i \tag{13.44}$$

Having obtained the above expression for V_c, the complex amplitude of the desired voltage, we have completed the third step of the impedance method. Amazingly, notice that we have arrived at the same result as in Equation 13.16 in a few easy steps.

Although not always necessary, we will proceed with the fourth step of the impedance method and obtain the actual voltage v_c as a function of time. We

can do so by substituting the magnitude and phase of V_c, and the frequency of our input into Equation 13.41 as follows:

$$\begin{aligned} v_c(t) &= |V_c| \cos(\omega_1 t + \angle V_c) \\ &= \frac{V_i}{\sqrt{1 + (\omega_1 RC)^2}} \cos\left(\omega_1 t + \tan^{-1} \frac{-\omega_1 RC}{1}\right) \end{aligned}$$

Not surprisingly, this expression for $v_c(t)$ is the same as that in Equation 13.23, but derived with significantly less effort.

As a final note, this is the forced response for a cosine wave drive. If the excitation is a tone burst, then the homogeneous solution must be added to obtain the complete solution.

13.3.1 Example: Series RL Circuit

Next, as a further illustration of the concept of impedance, let us find the voltage across the resistor in the RL circuit of Figure 13.11, assuming that $v_i = V_i \cos \omega_1 t$. Figure 13.11b shows the corresponding impedance model.

The generalized voltage divider relation for V_o in the impedance model, Figure 13.11b, is

$$V_o = \frac{Z_R}{Z_R + Z_L} V_i \tag{13.45}$$

where Z_R and Z_L are the impedances of the resistor and the inductor respectively. Substituting the actual impedance values, we obtain

$$V_o = \frac{R}{R + Ls} V_i \tag{13.46}$$

Recall from Equation 13.10 that we have been using s as a shorthand for $j\omega$. So at any frequency, ω_1,

$$V_o = \frac{R}{R + j\omega_1 L} V_i \tag{13.47}$$

The denominator on the right hand side of Equation 13.47 is the impedance Z seen by the voltage source at the frequency ω_1. In other words,

$$Z(j\omega_1) = R + j\omega_1 L$$

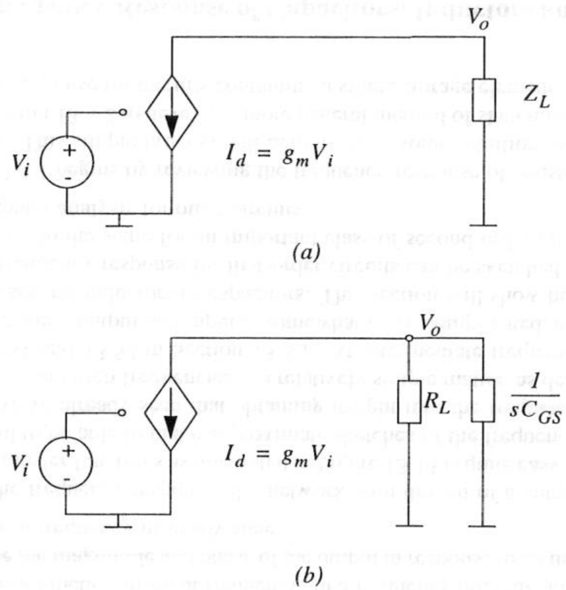


Figure 13.21: Impedance model

Substituting the polar form of $V_o(j\omega)$ into Equation 13.76 we get

$$v_o = Re\left[-g_m \frac{R_L}{\sqrt{1+(\omega R_L C_{GS})^2}} V_i e^{j(\omega t+\phi)}\right] \tag{13.77}$$

$$= -g_m \frac{R_L}{\sqrt{1+(\omega R_L C_{GS})^2}} V_i \cos(\omega t+\phi) \tag{13.78}$$

where $\phi = \tan^{-1}(-\omega R_L C_{GS})$.

Let us analyze the response at various frequencies. Equation 13.78 shows that at low frequencies ($\omega \to 0$) the expression for v_o is no different from that in Equation 13.74. Thus the response to a DC signal or a very low frequency sinusoid is similar to the response when the capacitor is absent. This is not surprising because the capacitor behaves as an open circuit at very low frequencies.

However, notice that as the frequency of the input sinusoid increases, the amplitude of the output sinusoid decreases. In fact, for very high frequencies ($\omega >> 1/R_L C_{GS}$), the amplitude of v_o tends to 0. The insight behind the high frequency result is that the capacitor behaves as a short for high frequencies. The resulting zero impedance of the load reduces the amplifier gain to 0.

13.4 Frequency Response: Magnitude and Phase versus Frequency

We characterize the behavior of a network by its *frequency response*.

> The frequency response is a plot of the magnitude and the phase of the network's transfer function as a function of frequency.

> The transfer function, which is also known as a *system function*, is the ratio of the complex amplitude of the network output to the complex amplitude of the input.

Equation 13.57 is one example of a system function, and Figure 13.13, a plot of the magnitude and phase of the system function versus frequency, is the frequency response.

The frequency response contains a lot of information about the system. It includes a magnitude plot and a phase plot, both as a function of frequency. The magnitude of the network's transfer function is the ratio of the amplitudes of the output and the input, and indicates the *gain* of the system as a function of frequency. The phase is the angular difference between the output and the input sinusoids.

Observing the frequency response of networks represents a major difference in perspective from the preceding chapters. The earlier chapters presented time-domain analyses in which our focus was on finding an output signal value as a function of time for a given input signal also specified as a function of time. For example, as shown in Figure 10.2, the step response of an RC network plotted the output voltage of an RC network as a function of time in response to a step input. In contrast, the frequency response represents a *frequency domain analysis* in which output behavior is presented as a function of input frequency. In a frequency domain analysis our goal is to determine the magnitude and phase of the output in response to an input sinusoidal signal of a given frequency in steady state.

Plotting the frequency response of a network with the aid of a computer for arbitrary system transfer functions as illustrated in Figure 13.13 is quite easy. Nevertheless, it is still useful to be able to make approximate sketches of the frequency response by inspection. We've already seen that obtaining insight into the frequency response at low frequencies and high frequencies is a relatively simple matter, as demonstrated by Equations 13.51 and 13.52 in Section 13.3.1. At intermediate frequencies, however, the relation between output and input is somewhat more complicated, especially for a network with several inductors or capacitors. This section will show how the general shape of the frequency response for first-order circuits can be sketched by inspection. Chapter 14 will do the same for an important class of second order circuits. We will resort to computer analysis for other circuits.[4]

Section 13.4.1 begins by reviewing the frequency response of resistors, capacitors and inductors. This simple process will help us build some intuition, which will then be used in Section 13.4.2 to develop a more general method of sketching by inspection the frequency response for circuits containing a single storage element and a resistor.

13.4.1 Frequency Response of Capacitors, Inductors and Resistors

Resistors, inductors, and capacitors result in transfer functions of the form s, $1/s$ or a constant. Recall from Section 13.3 that the element laws for resistors, inductors and capacitors in terms of complex amplitudes of voltages and currents are given by

$$\begin{aligned}\text{Resistor:}\quad V_o &= RI_o\\ \text{Inductor:}\quad V_o &= sLI_o = j\omega LI_o\end{aligned}$$

[4]There do exist methods for sketching frequency response plots for arbitrary circuits without the use of a computer. One of these, the Bode plot method, is discussed in Section 13.4.3. The popularity of these methods, however, has waned in recent times due to the widespread use of computers.

$$\text{Capacitor:}\quad V_o = \frac{1}{sC}I_o = \frac{1}{j\omega C}I_o$$

respectively. We can rewrite the above expressions in the form of V_o/I_o transfer functions that relate the complex voltage amplitudes to the complex current amplitudes as a function of frequency as follows:

$$\begin{aligned}\text{Resistor:}\quad \frac{V_o}{I_o} &= H(j\omega) = R\\ \text{Inductor:}\quad \frac{V_o}{I_o} &= H(j\omega) = j\omega L\\ \text{Capacitor:}\quad \frac{V_o}{I_o} &= H(j\omega) = \frac{1}{j\omega C}\end{aligned}$$

Here each of the transfer functions is an impedance. The above transfer functions are complex numbers. They are also frequency dependent. The corresponding magnitudes and phases are given by

$$\begin{aligned}\text{Resistor:}\quad \left|\frac{V_o}{I_o}\right| &= R \text{ and } \angle\frac{V_o}{I_o} = 0\\ \text{Inductor:}\quad \left|\frac{V_o}{I_o}\right| &= \omega L \text{ and } \angle\frac{V_o}{I_o} = 90^\circ\\ \text{Capacitor:}\quad \left|\frac{V_o}{I_o}\right| &= \frac{1}{\omega C} \text{ and } \angle\frac{V_o}{I_o} = -90^\circ\end{aligned}$$

Figure 13.22 plots the frequency response for the resistor, inductor and the capacitor using these values for the elements.

$$\begin{aligned}R &= 1\Omega\\ L &= 1\mu H\\ C &= 1\mu C\end{aligned}$$

As shown in the figure, the frequency response is a plot of the magnitude and phase of the transfer function versus frequency.

Frequency response plots are commonly plotted using log scales. As we will see shortly, log scales make the magnitudes of the responses due to capacitors and inductors appear as straight lines in the graph. Log scales also allow us to observe the response over many orders of magnitude variation in the frequency without necessarily compressing the low frequency behavior of the response (near zero radians/sec) into

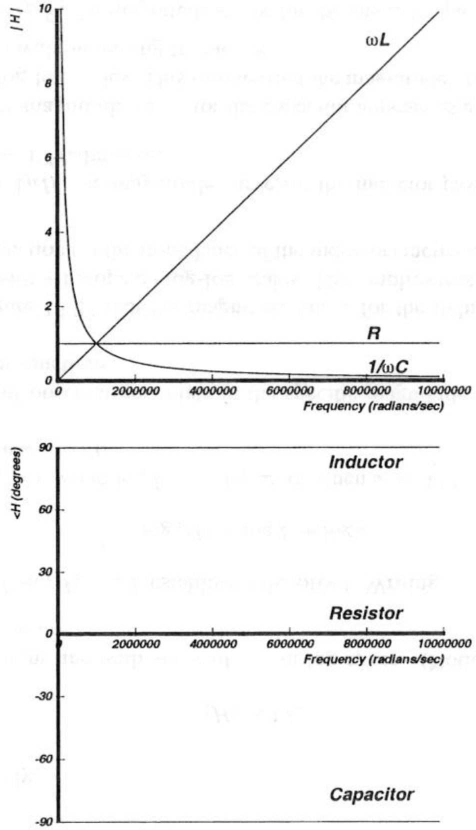


Figure 13.22: Frequency response of inductors, capacitors, and resistors plotted on linear scales

the magnitude and phase axes. The frequency responses for the resistor, inductor and the capacitor can be sketched on log scales by using the following relations.[5]

$$
\begin{aligned}
\text{Resistor:} \quad \log\left|\frac{V_o}{I_o}\right| &= \log R \\
\text{Inductor:} \quad \log\left|\frac{V_o}{I_o}\right| &= \log \omega L = \log L + \log \omega \\
\text{Capacitor:} \quad \log\left|\frac{V_o}{I_o}\right| &= \log \frac{1}{\omega C} = -\log C - \log \omega
\end{aligned}
$$

Figure 13.23 shows the corresponding plots. We can make several observations about the plots. The first set of observations relate to the nature of logarithmic plots in general.

- If x is the variable being plotted using a log scale, x is ***multiplied*** by a fixed factor for each fixed length increment along the axis. In contrast, on a linear scale, x is incremented by a fixed amount for each fixed length increment along the axis. For example, equal length increments along the frequency axis on a linear scale might correspond to the values 0, 2×10^6, 4×10^6, 6×10^6, 8×10^6, and so on. On the other hand, equal length increments along the frequency axis on a log scale might correspond to the values 10^4, 10^5, 10^6, 10^7, and so on.[6]
- There are two equivalent ways of plotting log functions:
  1. Plot $\log x$ on a linear scale.
  2. Plot x on a logarithmic scale.

  In Figure 13.23, we have plotted the magnitude functions both ways. In other words, for the abscissa, we plot both $\log \omega$ on a linear scale and ω on a logarithmic scale. For the ordinate, R, ωL and $1/\omega C$ are also plotted both ways. In the future, we will choose to plot x on a logarithmic scale.[7]

  On the phase plot, the horizontal scale is logarithmic, and the vertical scale is linear.

[5] In deriving the log relations here and in the future, we assume that both the left and right hand sides of the equations are divided by appropriate unit constants before taking the logs so that the arguments to the log functions are unitless.

[6] Historically, an octave is used to indicate a 2x change in frequency. For example, 2kHz is a 1 octave increase in frequency from 1kHz. Similarly, 4kHz is a 2 octave increase from 1kHz, and 8kHz is a 3 octave increase from 1kHz. In like manner, 500Hz is 1 octave below 1kHz.

Another useful term is a decade. A decade is a range of frequencies in which the highest frequency is 10x the lowest frequency. For example, the range from 1kHz to 10kHz is 1 decade, and that from 1kHz to 100kHz is 2 decades.

[7] In the literature it is also common to plot the response magnitude in decibels (dB), defined as

- The function

$$|H| = \omega$$

plots as a straight line with slope of +1 in log space, given consistent horizontal and vertical scales, because it changes by a factor of 10 for a factor of 10 increase in ω.

- Correspondingly,

$$|H| = 1/\omega$$

plots as a straight line with slope of -1 in log space. (Notice that $log|H| = \log 1/\omega = -\log \omega$.)

- The value of L in $|H| = \omega L$ establishes the offset. Writing

$$\log|H| = \log L + \log \omega$$

Thus, $\log|H| = 0$ when $\log L = -\log \omega$, or when $\omega = 1/L$. Put another way, $|H| = 1$, when $\omega = 1/L$.

The second set of observations relate to the specific magnitude and phase curves for our three transfer functions.

- Notice in Figure 13.23 that the magnitude curve for the inductor appears as a straight line with +1 slope on log-log scales. This implies that the magnitude of the transfer function (or the impedance of the inductor) increases with increasing frequency.

  Because $L = 1\mu H$, the magnitude curve for the inductor passes through unity for $\omega = 1/L = 10^6$radians/sec.

- In contrast, the magnitude curve for the capacitor appears as a straight line with -1 slope on log-log scales. This implies that the magnitude of the transfer function decreases with increasing frequency.

  Because $C = 1\mu F$, the magnitude curve for the capacitor passes through unity for $\omega = 1/C = 10^6$radians/sec.

- The magnitude plot for the resistor appears as a horizontal line.

Response ratio in $dB = 20 \log_{10}|H(j\omega)|$	(13.79)

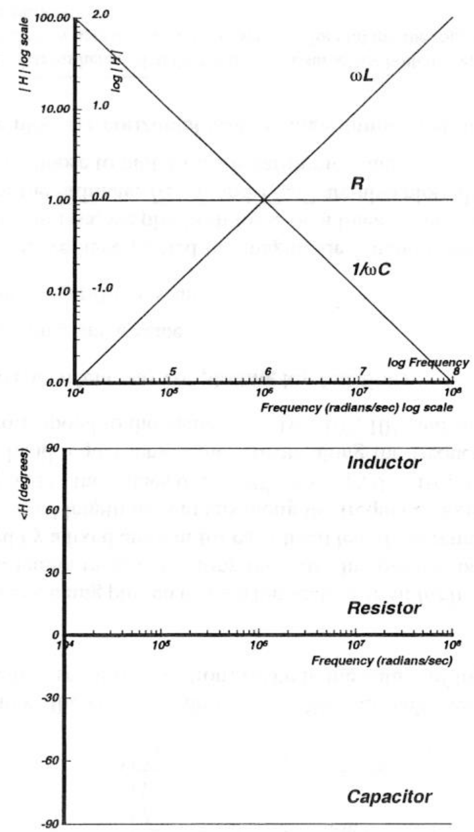


Figure 13.23: Frequency response of inductors, capacitors, and resistors plotted on log scales

- The phase plot for the inductor shows that the inductor causes a fixed phase shift of 90°, while that for the capacitor indicates a phase shift of −90°. The resistor does not introduce any phase shift.

As we will see next, these simple plots for resistors, capacitors and inductors provide much of the necessary insight to plot the frequency responses for circuits containing a resistor and a storage element.

13.4.2 Intuitively Sketching the Frequency Response of RC and RL Circuits

Let us now examine the frequency response of circuits with a single storage element and a single resistor, an important class of first order circuits, and see how their responses can be sketched by inspection. Such circuits result in transfer functions of the form $1/(s+a)$, $(s+a)$, $s/(s+a)$, $(s+a)/s$, where a is some constant. We will illustrate the approach using as an example the series RL circuit from Figure 13.11 in Section 13.3.1.

The input-output relationship as a function of the input drive frequency ω for the series RL circuit of Figure 13.11 is given by

$$V_o = \frac{R}{R+sL}V_i \qquad (13.80)$$

(see Equation 13.47). Dividing by V_i, we obtain its transfer function:

$$H(j\omega) = \frac{V_o}{V_i} = \frac{R}{R+sL} \qquad (13.81)$$

For reasons that will be clear shortly, let's rewrite this in a more standard form as

$$H(j\omega) = \frac{V_o}{V_i} = \frac{R/L}{R/L+s} \qquad (13.82)$$

The magnitude of the transfer function is

$$|H(j\omega)| = \left|\frac{R/L}{R/L+j\omega}\right| \qquad (13.83)$$

and its phase is

$$\angle H(j\omega) = \tan^{-1}\frac{-\omega L}{R} \qquad (13.84)$$

This frequency response using logarithmic scales for the horizontal and vertical axes was previously shown in Figure 13.13. We repeat here in Figure 13.24 a computer

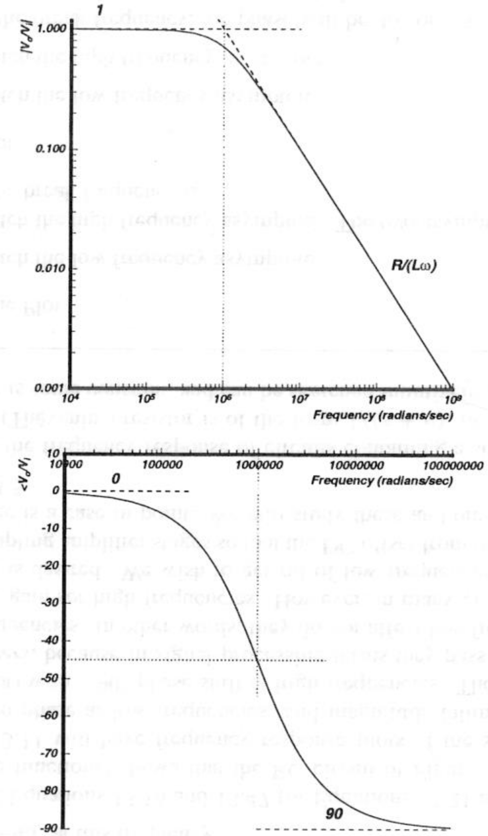


Figure 13.24: Magnitude and phase of V_o/V_i versus frequency ω on log scales

generated plot of the same response plot (assuming, as before, $L = 1mH$, $R = 1k\Omega$, so that $R/L = 10^6$ radians/second).

The same frequency response can also be sketched easily by making the following observations about the magnitude and phase plots. Let us first deal with the magnitude plot. Observe from Figure 13.24 that the magnitude plot is asymptotic to two straight lines. At low frequencies ($\omega \to 0$), the magnitude becomes

$$|H(j\omega)| \simeq 1 \tag{13.85}$$

Thus, at low frequencies the magnitude is unity, and hence appears as a horizontal line.

At high frequencies ($\omega \to \infty$), the ω term dominates the expression in the denominator and the magnitude becomes

$$|H(j\omega)| = \frac{R/L}{\omega} \tag{13.86}$$

From our observations on log plots in Section 13.4.1, we know that the log plot for the expression in Equation 13.86 will appear as a straight line with a slope of -1 for consistent horizontal and vertical scales, and passes through the point $|H| = 1$ when $\omega = R/L$.

Clearly the two asymptotes intersect at

$$\omega = \frac{R}{L} = 10^6 \text{ radians/second}$$

called the *break frequency* or corner frequency. At the break frequency, the true magnitude of $H(j\omega)$ is

$$|H(j\omega)| = |10^6/(10^6 + j10^6)| = \frac{1}{\sqrt{2}} = 0.707$$

Thus the break frequency is also called the 0.707 frequency. At this frequency, the real and imaginary parts of the function are equal.[8]

As is clear from Figure 13.24, the high and low frequency asymptotes approximate the frequency response curve pretty well.

Let us now address the phase plot. Like the magnitude plot, the phase curve can also be approximated by the low and high frequency asymptotes. At low frequencies the phase becomes

$$\angle H(j\omega) \simeq 0 \tag{13.87}$$

[8]Since 0.707 in decibels is $20 \log 0.707 = -3$dB, the break frequency is also called a -3dB frequency. Since $0.707^2 = 0.5$, the break frequency is also called the half power point. When the magnitude curve begins to dip after the break frequency, the break frequency is also called the cutoff frequency.

while at high frequencies the phase is

$$\angle H(j\omega) = -90^\circ \tag{13.88}$$

Notice that the phase curve goes smoothly from 0° at $\omega = 0$ to -90° at $\omega = \infty$. At the break point, the real and imaginary parts of Equation 13.83 are equal, hence the angle of $H(j\omega)$ is -45° at this frequency.

Examination of Equations 13.16 and 13.47 (or Equations 13.21 and 13.50 for the corresponding time functions) shows that the RC circuit of Figure 13.3 and the RL circuit of Figure 13.11 will have frequency response plots of the same form: unit magnitude and zero phase at low frequencies, and magnitude falling as $1/\omega$ (slope of -1 in log-log plot) with -90° phase shift at high frequencies. These functions are called *low-pass filters*, because in signal-processing terms they pass low frequencies and reject high frequencies. In other words, they do not affect low frequencies, while they provide a low gain for high frequencies. However, in many circuit applications the opposite effect is desired. We wish to get rid of low frequencies, and pass high frequencies. Decoupling amplifier stages so that the DC offset from one stage does not affect the next stage is a case in point. We will study these and other filters in more detail in Section 13.5.

To summarize, the frequency response of circuits containing a single storage element and a single (Thévenin) resistor is of the form $1/(s+a)$, $(s+a)$, $s/(s+a)$, $(s+a)/s$, where a is some constant, and can be sketched intuitively as follows:

- Magnitude Plot
  1. Sketch the low frequency asymptote
  2. Sketch the high frequency asymptote. The two asymptotes intersect at the break frequency a.
- Phase Plot
  1. Sketch the low frequency asymptote
  2. Sketch the high frequency asymptote
  3. At the break frequency, the phase will be 45° or -45° as appropriate. Draw a smooth line starting with the low frequency asymptote, passing through 45° or -45° as appropriate at the break frequency, and finishing off at the high frequency asymptote.

Example 13.4 Intuitive Sketch of the Frequency Response of RC Circuit

Let us sketch the frequency response for the transfer function relating V_r to V_i for the RC circuit shown in Figure 13.14.

From Equation 13.58, we can immediately write the transfer function as

$$\frac{V_r}{V_i} = H(s) = \frac{500 \times 10^3}{500 \times 10^3 + \frac{1}{1\times 10^{-9}s}} \tag{13.89}$$

Simplifying,

$$H(s) = \frac{s}{s + 2000}$$

Substituting, $s = j\omega$,

$$H(j\omega) = \frac{j\omega}{j\omega + 2000} \tag{13.90}$$

Since the transfer function is of the form $s/(s + a)$, we can apply our intuitive method for sketching the frequency response.

- Magnitude Plot
  1. Sketch the low frequency asymptote:
     The low frequency asymptote ($\omega << 2000$) is given by
     $$|H| = \frac{\omega}{2000}$$
  2. Sketch the high frequency asymptote. The two asymptotes intersect at the break frequency.
     The high frequency asymptote ($\omega >> 2000$) is given by
     $$|H| = 1$$
     The two asymptotes intersect at the break frequency
     $$\omega = 2000 \text{ radians/sec}$$
     The dashed lines in the magnitude plot in Figure 13.25 shows these two asymptotes. They intersect at $\omega = 2000$ radians/sec. A computer generated plot of the magnitude versus frequency is also shown. Together, the asymptotes are a fairly good approximation of the magnitude curve.
- Phase Plot
  1. Sketch the low frequency asymptote
     The low frequency asymptote for the phase is given by
     $$\angle H = 90°$$

  2. Sketch the high frequency asymptote
     The high frequency asymptote for the phase is given by
     $$\angle H = 0°$$
     The dashed lines in the phase plot in Figure 13.25 show these two asymptotes.
  3. The point $(2000, 45°)$ denoting the phase at the break frequency is also marked. The true phase curve is also shown, starting with the low frequency asymptote, passing through 45° at the break frequency, and finishing off at the high frequency asymptote.

13.4.3 The Bode Plot: Sketching the Frequency Response of General Functions *

Sections 13.4.1 and 13.4.2 demonstrated the ease with which we can sketch the frequency response of simple circuits by observing their behavior at low frequencies and high frequencies. Things get more complicated for a network with several inductors or capacitors. This section discusses a simple and intuitive method called Bode plots for sketching the frequency response of more general circuits. The Bode method uses the insight gained from Sections 13.4.1 and 13.4.2 that the frequency response plots can be closely approximated by straight line segments derived from the asymptotic behavior of the transfer functions.

The method proceeds as follows. First, write the relationship (Equation 13.46, for example) in the form of a system function, the ratio of the complex amplitude of the response to the complex amplitude of the input:

$$H(s) = \frac{\text{Response}}{\text{Input}} \tag{13.91}$$

In general, the system function $H(s)$ will be the ratio of two polynomials:

$$H(s) = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \tag{13.92}$$

where the coefficients a_i and b_i are real numbers since our circuit parameters are real numbers. We saw one example of this in Equation 13.64. We can factor the numerator and denominator polynomials and write:

$$H(s) = \frac{K_1(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \tag{13.93}$$

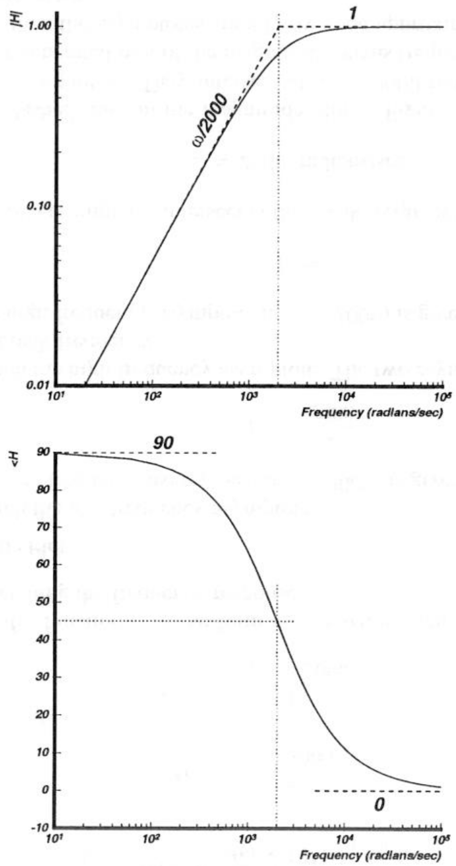


Figure 13.25: Frequency response

where K_1 is a constant and $z_1, z_2, \cdots, z_m$ are the roots of the numerator polynomial, and $p_1, p_2, \cdots, p_n$ are the roots of the denominator polynomial.[9]

In general, some of the roots of the numerator or the denominator polynomials can be zero. Furthermore, the roots of the numerator and denominator polynomials can also be complex. If any of the roots are complex, then they must appear in complex conjugate pairs, so that the overall system function remains real. We will rewrite Equation 13.93 into the following standard form to reflect these facts.

$$H(s) = \frac{K_o s^l (s+a_1)(s+a_2)\cdots(s^2+2\alpha_1 s+\omega_1^2)\cdots}{(s+a_3)(s+a_4)\cdots(s^2+2\alpha_2 s+\omega_2^2)\cdots} \tag{13.94}$$

In Equation 13.94, we have combined complex conjugate pairs into quadratic terms of the form $(s^2 + 2\alpha s + \omega^2)$. Thus all the remaining a_i values are real. The s^l term, where l can be positive or negative, reflects the case where the roots in Equation 13.93 are zero.

We will now show that it is possible to sketch without formal calculation the general shape of $H(s)$ as a function of frequency. More precisely, we can make an approximate sketch of the magnitude and phase of $H(s)$ as a function of the input frequency ω. The resulting pair of graphics representing an approximate sketch of the frequency response is called a *Bode plot*, in honor of the Bell Laboratories engineer who devised it to study stability in feedback amplifiers.[10]

The Bode plot is an approximation of the frequency response and accordingly has two parts: a sketch of the log magnitude of $H(j\omega)$ versus $\log\omega$ and a sketch of the angle of $H(j\omega)$ versus $\log\omega$. These coordinates are chosen because they facilitate straightforward construction of the frequency response graphs even for complicated functions without the use of a computer. Taking the magnitude and log on both sides of Equation 13.94,

$$\begin{aligned} log|H(s)| &= \log K_o + \\ &\quad \log|s| + \log|s| + \cdots \ (l \text{ terms}) + \\ &\quad \log|s+a_1| + \log|s+a_2| + \cdots - \log|s+a_3| - \log|s+a_4| + \cdots \end{aligned}$$

[9]Because the system function goes to zero when $s = z_i$, the roots of the numerator, $z_1, z_2, \cdots, z_m$, are called the *zeros*, *definition* of the system function. Similarly, the roots of the denominator, $p_1, p_2, \cdots, p_n$ are called the *poles* of the system function. The system function goes to infinity when s takes on the value of one of the poles (in other words, when $s = p_i$). When one or more of the $z_i's$ or $p_i's$ is zero, the system is said to have zeros or poles at the origin. The poles and zeros of a system function are important system parameters because they characterize the general behavior of the system. A detailed discussion of system analysis using poles and zeros is beyond the scope of this book.

[10]Bode, H.W., *Network Analysis and Feedback Amplifier Designs*, Van Nostrand, New York, 1945, Chapter 15.

$$\log|s^2 + 2\alpha_1 s + \omega_1^2| + \cdots - \log|s^2 + 2\alpha_2 s + \omega_2^2| + \cdots \quad (13.95)$$

and for the phase

$$\begin{aligned} \angle H(s) &= \angle K_o + \\ & \angle s + \angle s + \cdots \ (l \text{ terms}) + \\ & \angle(s + a_1) + \angle(s + a_2) + \cdots - \angle(s + a_3) - \angle(s + a_4) - \cdots \\ & + \angle(s^2 + 2\alpha_1 s + \omega_1^2) + \cdots - \angle(s^2 + 2\alpha_2 s + \omega_2^2) - \cdots \end{aligned} \quad (13.96)$$

Notice that there are four types of terms in the magnitude and phase equations.

1. The K_o constant term,
2. the s terms,
3. terms of the form $(s + a)$, and
4. quadratic terms of the form $(s^2 + 2\alpha s + \omega^2)$, which have complex roots.

This gives us a simple way of approximating the magnitude and phase curves of the frequency response plot. First, draw the individual magnitude and angle curves for each of the four types of terms in the numerator and denominator of Equation 13.94. Then, construct the overall magnitude and phase plots by simply adding together the individual curves.

Let us now address each of the four terms.

1. The K_o constant term

   We saw how to draw the frequency response of constant terms in Section 13.4.1. Essentially constant terms result in horizontal lines on the magnitude plot and have a phase of zero.

2. The s terms

   Terms of the form s and $1/s$ (if l is negative) were also plotted in Section 13.4.1. We saw that each of these terms result in lines of $+1$ or -1 slope on the log magnitude plot and contribute to a phase of 90° or $-90°$ respectively.

3. Terms of the form $(s + a)$

   Section 13.4.2 addressed terms of the form $(s + a)$. We showed that the magnitude part of the frequency response of these terms is approximated by two straight lines corresponding to the low and high frequency asymptotes meeting at the break frequency a. Accordingly, Bode plots result in a series of straight line segments attached together at the break frequencies.

   The phase plot also uses low and high frequency asymptotes and passes through 45° at the break frequency a. For more accuracy, the phase curve can be approximated by a straight line that passes through 45° at the break frequency a, and meets the low and high frequency asymptotes at 0.1 times the break frequency $(0.1a)$ and 10 times the break frequency $(10a)$ respectively.

4. Quadratic terms of the form $(s^2 + 2\alpha s + \omega^2)$ with complex roots

   Although not as straightforward, it is possible to sketch frequency response plots for system functions of the form $(s^2 + 2\alpha s + \omega^2)$, where the roots are complex. However, we will defer a further discussion on plotting Bode plots for complex roots to Section 14.4. For now, we will focus on real roots.

Example 13.5 Bode Plot for Series RL Circuit

Let us sketch the Bode plot for the RL circuit of Figure 13.11. From Equation 13.47, the system function here is a voltage ratio

$$H(j\omega) = \frac{V_o}{V_i} = \frac{R/L}{j\omega + R/L} \quad (13.97)$$

To make the example specific, let us assume that the time constant L/R has a value of 50 msec. Thus the break frequency is at

$$a = \frac{R}{L} = 20 \ \text{rads/sec}$$

and the system function becomes

$$H(j\omega) = \frac{20}{j\omega + 20} \quad (13.98)$$

The system function has two terms: a constant term and a term of the form $(s + a)$. Figures 13.26 and 13.27 show the construction of the magnitude and phase plots respectively. The dashed lines in Figures 13.26c and 13.27c form the composite Bode plot, and are obtained by simple subtraction of (b) from (a). For reference, the solid lines show the true magnitude and phase functions. Note that at the break frequency, the true magnitude is given by

$$\begin{aligned} |H(j\omega)| &= 1/1.41 \\ &= 0.707 \end{aligned} \quad (13.99)$$

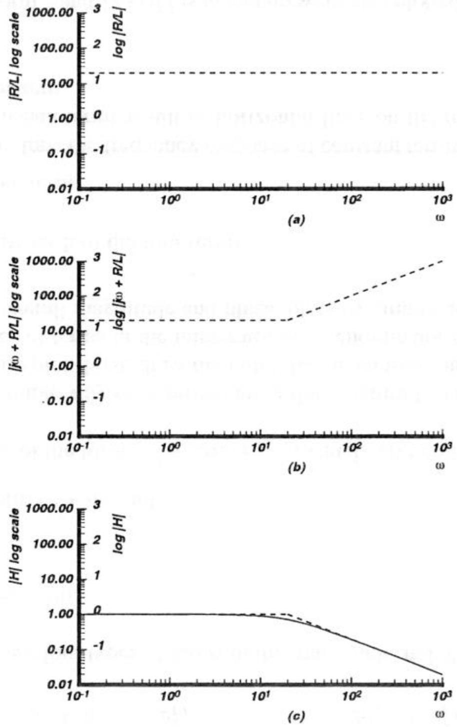


Figure 13.26: Magnitude curve of the Bode plot for RL circuit. (a) The magnitude curve for R/L. (b) The magnitude curve for $j\omega + R/L$. (c) The composite magnitude curve obtained by subtracting (b) from (a).

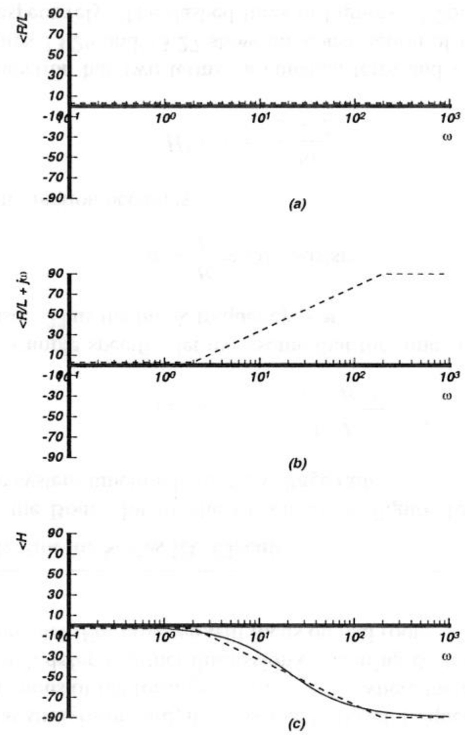


Figure 13.27: Phase curve of the Bode plot for RL circuit. (a) The phase curve curve for R/L. (b) The phase curve for $j\omega + R/L$. (c) The composite phase curve obtained by subtracting (b) from (a).

The principal advantage of the Bode plot is that the composite magnitude asymptotes for system functions that can be written in the form of Equation 13.94 are always lines of integer slope in log space. Further, any system function that can be written as a ratio of polynomials in ω (regardless of whether the roots are real or complex) must approach at both low and high frequencies $(j\omega)^n$, where n is some integer. Hence the magnitude asymptotes on Bode plots for both small and large ω must be straight lines of integer slope in log space, and the phase must approach a multiple of 90°.

Example 13.6 Another Bode Plot Example

To illustrate the Bode method for more general transfer functions, let us sketch the Bode plot for this transfer function.

$$H(j\omega) = \frac{0.025(1000 + j\omega)}{100 + j\omega} \tag{13.100}$$

The specific circuit that results in the above transfer function is not relevant to us right now, but will be discussed later in Section 13.6.

The system function has three terms: a constant term, and two terms of the form $(s+a)$. The Bode construction of the magnitude curve of the frequency response for the above transfer function is shown in Figure 13.28. The corresponding phase construction is shown in Figure 13.29. For reference, the actual frequency response generated using a computer is shown using solid curves.

13.5 Filters

The frequency response of several of the circuits considered in the previous sections indicated their frequency selective behavior (for example, the RL circuit in Figure 13.11 or the RC circuit in Figure 13.15). We can use such circuits to process signals according to their frequency. Circuits used in this manner are called *filters*. Filters are a major application of frequency domain analysis. The signal-processing property of filtering is fundamental to the operation of all television, radio, and cellular phone receivers, which must select one transmitted signal from among many present at the receiver antenna.

The frequency response plots (see Figure 13.24) of the RL circuit in Figure 13.11 shows that it rejects (i.e., attenuates) high frequencies and passes (i.e., does not affect) low frequencies, and therefore behaves like a *low pass filter*. The RC circuit in Figure 13.15 behaves like a *band pass filter* because it passes frequencies that fall within

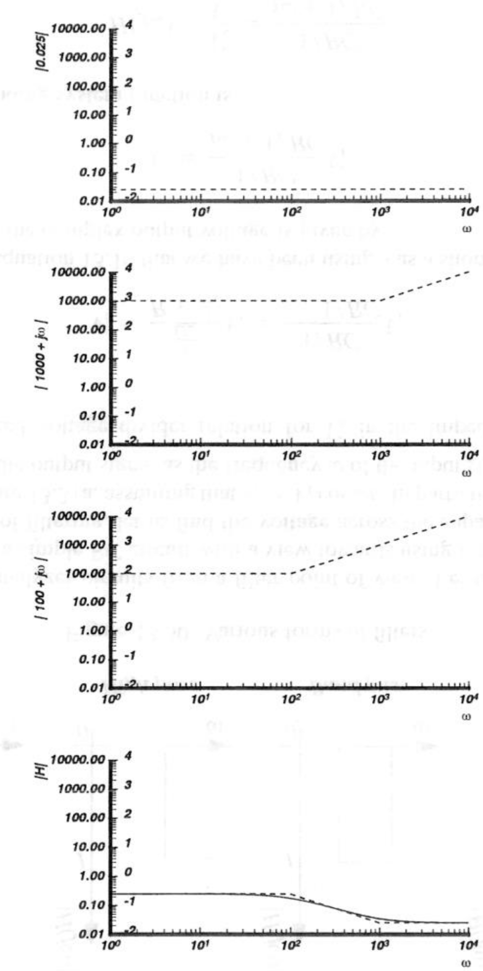


Figure 13.28: Construction of the magnitude curve of the Bode plot. The composite magnitude curve for the transfer function is obtained by subtracting the magnitude curve of $(1000 + j\omega)$ from the sum of the magnitude curves of 0.025 and $(100 + j\omega)$.

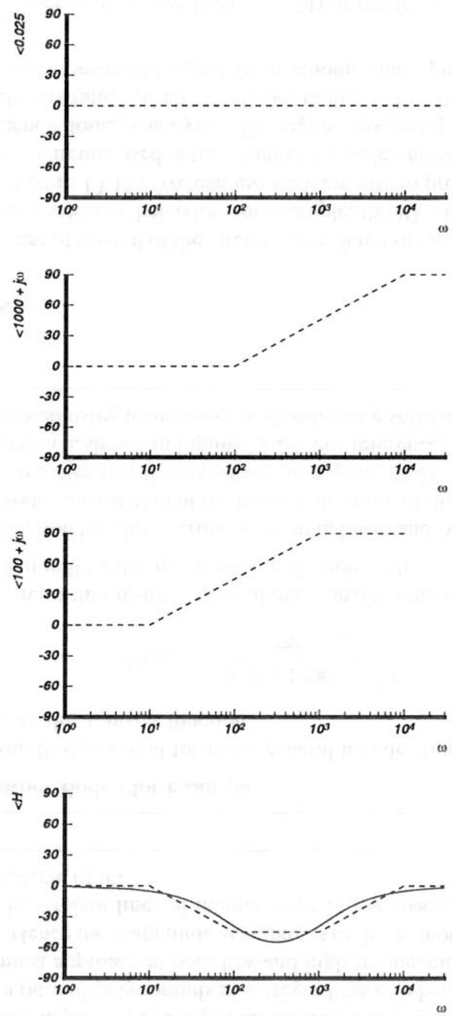


Figure 13.29: Construction of the phase curve of the Bode plot.

a certain band and rejects both very low frequencies and high frequencies (see Figure 13.17). In general, we can build many other types of filters as well. Figure 13.30 shows in abstract form the magnitude curves of the frequency response for several types of filters.

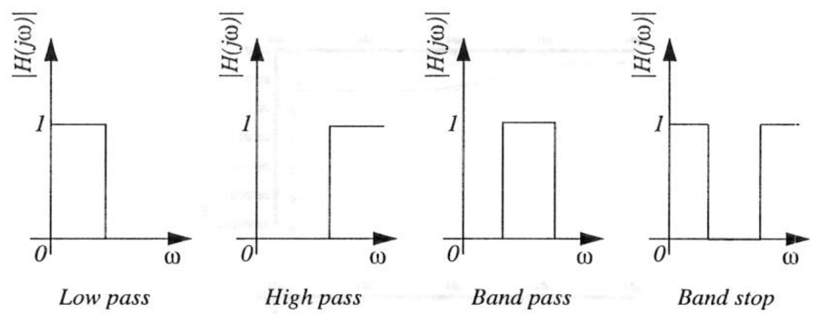


Figure 13.30: Various forms of filters

This section analyzes circuits from a filter point of view. Let us begin with a detailed analysis of a simple RC circuit with a view towards using it as a filter. To illustrate the concept of filtering, let us find the voltage across the capacitor in the simple RC circuit of Figure 13.31a, assuming that $v_i = V_i \cos \omega t$. In particular, we wish to find the amplitude of the output signal as the frequency ω of the input signal is changed.

The generalized voltage divider relation for V_o in the impedance model, Figure 13.31b, is

$$V_o = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_i = \frac{1/RC}{s + 1/RC} V_i \qquad (13.101)$$

Recall from Equation 13.10 that we have been using s as a shorthand for $j\omega$. So at any frequency, ω, the complex output voltage is given by

$$V_o = \frac{1/RC}{j\omega + 1/RC} V_i \qquad (13.102)$$

The corresponding system function is

$$H(j\omega) = \frac{V_o}{V_i} = \frac{1/RC}{j\omega + 1/RC} \qquad (13.103)$$

It is easy to see that the magnitude of the system function at low frequencies is close to unity. At high frequencies, on the other hand, the magnitude of the system

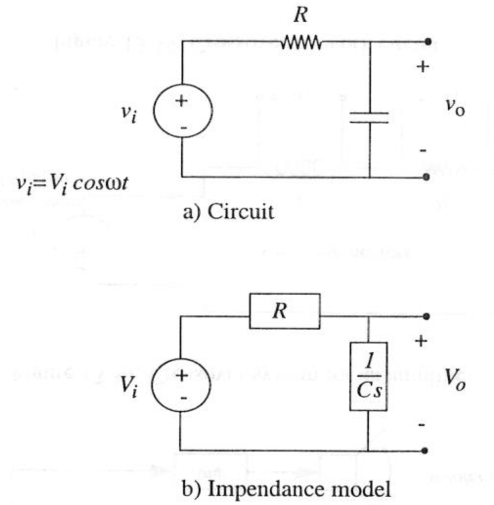


Figure 13.31: A simple RC filter circuit and its impedance model

function approaches 0. Because low frequencies are passed and high frequencies are rejected or attenuated, this circuit acts as a low pass filter.

Figure 13.32 plots the frequency response of the RC circuit assuming that the RC time constant of the circuit is $RC = \frac{1}{20}$ seconds. The shape of the magnitude plot is indicative of a low pass filter. The break frequency is 20 rads/sec. This says that the filter begins to reject input signals whose frequencies are in the vicinity of 20 rads/sec. The level of attenuation increases as the frequency increases beyond the break frequency.

Frequency cutoff begins in the vicinity of the break frequency. Hence the break frequency is also called the cutoff frequency. Thus we can design our RC low pass filter to have any cutoff frequency by an appropriate choice of the RC time constant. As pictured in Figure 13.33, the higher the value of RC the lower the cutoff frequency of the filter.

Finally, noting the similar forms of Equations 13.75 and 13.103, we conclude that Figure 13.32 reflects the frequency response of the circuit in Figure 13.21 as well. In fact, the circuit in Figure 13.21 is a Norton version of the circuit in Figure 13.31.

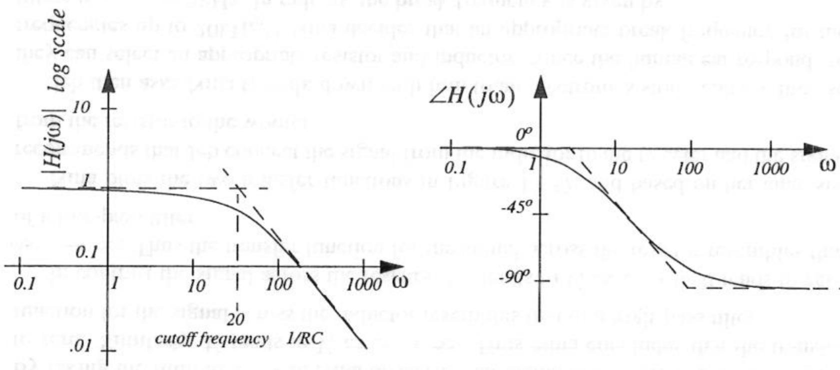


Figure 13.32: Frequency response of the simple RC filter circuit

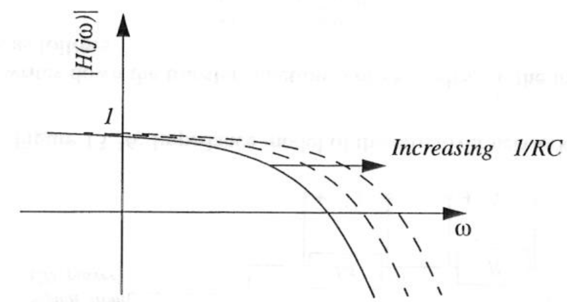


Figure 13.33: Designing the cutoff frequency (or break frequency) of a filter