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Preface

Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my "class notes" they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I've tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.
2. Because I want these notes to provide some more examples for you to read through, I don't always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.
3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.
4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Systems of Differential Equations

Introduction

To this point we've only looked as solving single differential equations. However, many "real life" situations are governed by a system of differential equations. Consider the population problems that we looked at back in the modeling section of the first order differential equations chapter. In these problems we looked only at a population of one species, yet the problem also contained some information about predators of the species. We assumed that any predation would be constant in these cases. However, in most cases the level of predation would also be dependent upon the population of the predator. So, to be more realistic we should also have a second differential equation that would give the population of the predators. Also note that the population of the predator would be, in some way, dependent upon the population of the prey as well. In other words, we would need to know something about one population to find the other population. So to find the population of either the prey or the predator we would need to solve a system of at least two differential equations.

The next topic of discussion is then how to solve systems of differential equations. However, before doing this we will first need to do a quick review of Linear Algebra. Much of what we will be doing in this chapter will be dependent upon topics from linear algebra. This review is not intended to completely teach you the subject of linear algebra, as that is a topic for a complete class. The quick review is intended to get you familiar enough with some of the basic topics that you will be able to do the work required once we get around to solving systems of differential equations.

Here is a brief listing of the topics covered in this chapter.

Review : Systems of Equations – The traditional starting point for a linear algebra class. We will use linear algebra techniques to solve a system of equations.

Review : Matrices and Vectors – A brief introduction to matrices and vectors. We will look at arithmetic involving matrices and vectors, inverse of a matrix, determinant of a matrix, linearly independent vectors and systems of equations revisited.

Review : Eigenvalues and Eigenvectors – Finding the eigenvalues and eigenvectors of a matrix. This topic will be key to solving systems of differential equations.

Systems of Differential Equations – Here we will look at some of the basics of systems of differential equations.

Solutions to Systems – We will take a look at what is involved in solving a system of differential equations.

Review : Systems of Equations

Because we are going to be working almost exclusively with systems of equations in which the number of unknowns equals the number of equations we will restrict our review to these kinds of systems.

All of what we will be doing here can be easily extended to systems with more unknowns than equations or more equations than unknowns if need be.

Let's start with the following system of *n* equations with the *n* unknowns, $x_1, x_2,…, x_n$.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\tag{1}
$$

Note that in the subscripts on the coefficients in this system, a_{ij}, the *i* corresponds to the equation that the coefficient is in and the *j* corresponds to the unknown that is multiplied by the coefficient.

To use linear algebra to solve this system we will first write down the **augmented matrix** for this system. An augmented matrix is really just the all the coefficients of the system and the numbers for the right side of the system written in matrix form. Here is the augmented matrix for this system.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{pmatrix}$$

To solve this system we will use elementary row operations (which we'll define these in a bit) to rewrite the augmented matrix in triangular form. The matrix will be in triangular form if all the entries below the main diagonal (the diagonal containing a_{11}, a_{22}, …,a_{nn}) are zeroes.

Once this is done we can recall that each row in the augmented matrix corresponds to an equation. We will then convert our new augmented matrix back to equations and at this point solving the system will become very easy.

Before working an example let's first define the elementary row operations. There are three of them.
1. Interchange two rows. This is exactly what it says. We will interchange row i with row j. The notation that we'll use to denote this operation is : $R_i \leftrightarrow R_j$
2. Multiply row i by a constant, c. This means that every entry in row i will get multiplied by the constant c. The notation for this operation is : cR_i
3. Add a multiply of row i to row j. In our heads we will multiply row i by an appropriate constant and then add the results to row j and put the new row back into row j leaving row i in the matrix unchanged. The notation for this operation is : $cR_i + R_j$

It's always a little easier to understand these operations if we see them in action. So, let's solve a couple of systems.

Example 1 Solve the following system of equations.
$$-2x_1 + x_2 - x_3 = 4$$
$$x_1 + 2x_2 + 3x_3 = 13$$
$$3x_1 + x_3 = -1$$

Solution
The first step is to write down the augmented matrix for this system. Don't forget that coefficients of terms that aren't present are zero.
$$\begin{pmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{pmatrix}$$
Now, we want the entries below the main diagonal to be zero. The main diagonal has been colored red so we can keep track of it during this first example. For reasons that will be apparent eventually we would prefer to get the main diagonal entries to all be ones as well.

We can get a one in the upper most spot by noticing that if we interchange the first and second row we will get a one in the uppermost spot for free. So let's do that.

$$\begin{pmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{pmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{pmatrix}$$

Now we need to get the last two entries (the -2 and 3) in the first column to be zero. We can do this using the third row operation. Note that if we take 2 times the first row and add it to the second row we will get a zero in the second entry in the first column and if we take -3 times the first row to the third row we will get the 3 to be a zero. We can do both of these operations at the same time so let's do that.

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{pmatrix} \begin{matrix} 2R_1+R_2 \\ -3R_1+R_3 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{pmatrix}$$

Before proceeding with the next step, let's make sure that you followed what we just did. Let's take a look at the first operation that we performed. This operation says to multiply an entry in row 1 by 2 and add this to the corresponding entry in row 2 then replace the old entry in row 2 with this new entry. The following are the four individual operations that we performed to do this.

$$2(1)+(-2)=0$$
$$2(2)+1=5$$
$$2(3)+(-1)=5$$
$$2(13)+4=30$$

Okay, the next step optional, but again is convenient to do. Technically, the 5 in the second column is okay to leave. However, it will make our life easier down the road if it is a 1. We can use the second row operation to take care of this. We can divide the whole row by 5. Doing this gives,

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{pmatrix} \begin{matrix} \frac{1}{5}R_2 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{pmatrix}$$

The next step is to then use the third row operation to make the -6 in the second column into a zero.

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{pmatrix} \begin{matrix} 6R_2+R_3 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{pmatrix}$$

Now, officially we are done, but again it's somewhat convenient to get all ones on the

main diagonal so we'll do one last step.

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

We can now convert back to equations.

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 13 \\ x_2 + x_3 &= 6 \\ x_3 &= 2 \end{aligned}$$

At this point the solving is quite easy. We get x_3 for free and once we get that we can plug this into the second equation and get x_2. We can then use the first equation to get x_1. Note as well that having 1's along the main diagonal helped somewhat with this process.

The solution to this system of equation is

$$x_1 = -1 \qquad x_2 = 4 \qquad x_3 = 2$$

The process used in this example is called **Gaussian Elimination**. Let's take a look at another example.

Example 2 Solve the following system of equations.

$$x_1 - 2x_2 + 3x_3 = -2$$
$$-x_1 + x_2 - 2x_3 = 3$$
$$2x_1 - x_2 + 3x_3 = 1$$

Solution
First write down the augmented matrix.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{pmatrix}$$

We won't put down as many words in working this example. Here's the work for this augmented matrix.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ -2R_1 + R_3 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & -3 & 5 \end{pmatrix}$$

$$\xrightarrow[\rightarrow]{-R_2} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -3 & 5 \end{pmatrix} \xrightarrow[\rightarrow]{-3R_2 + R_3} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

We won't go any farther in this example. Let's go back to equations to see why.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix} \Rightarrow \quad \begin{array}{r} x_1 - 2x_2 + 3x_3 = -2 \\ x_2 - x_3 = -1 \\ 0 = 8 \end{array}$$

The last equation should cause some concern. There's one of three options here. First, we've somehow managed to prove that 0 equals 8 and we know that's not possible. Second, we've made a mistake, but after going back over our work it doesn't appear that we have made a mistake.

This leaves the third option. When we get something like the third equation that simply doesn't make sense we immediately know that there is no solution. In other words, there is no set of three numbers that will make all three of the equations true at the same time.

Let's work another example. We are going to get the system for this new example by making a very small change to the system from the previous example.

Example 3 Solve the following system of equations.
$$\begin{array}{r} x_1 - 2x_2 + 3x_3 = -2 \\ -x_1 + x_2 - 2x_3 = 3 \\ 2x_1 - x_2 + 3x_3 = -7 \end{array}$$

Solution
So, the only difference between this system and the system from the second example is we changed the 1 on the right side of the equal sign in the third equation to a -7.

Now write down the augmented matrix for this system.
$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & -7 \end{pmatrix}$$

The steps for this problem are identical to the steps for the second problem so we won't write them all down. Upon performing the same steps we arrive at the following matrix.
$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This time the last equation reduces to
$$0 = 0$$
and unlike the second example this is not a problem. Zero does in fact equal zero!

We could stop here and go back to equations to get a solution and there is a solution in this case. However, if we go one more step and get a zero above the one in the second column as well as below it our life will be a little simpler. Doing this gives,

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} 2R_2 + R_1 \\ \Rightarrow \end{array} \begin{pmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If we now go back to equation we get the following two equations.

$$\begin{pmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 + x_3 = -4 \\ \Rightarrow \qquad x_2 - x_3 = -1 \end{array}$$

We have two equations and three unknowns. This means that we can solve for two of the variables in terms of the remaining variable. Since x_3 is in both equations we will solve in terms of that.

$$x_1 = -x_3 - 4$$
$$x_2 = x_3 - 1$$

What this solution means is that we can pick the value of x_3 to be anything that we'd like and then find values of x_1 and x_2. In these cases we typically write the solution as follows,

$$x_1 = -t - 4$$
$$x_2 = t - 1 \qquad\qquad t = \text{ any real number}$$
$$x_3 = t$$

In this way we get an infinite number of solutions, one for each and every value of t.

These three examples lead us to a nice fact about systems of equations.

Fact

Given a system of equations, (1), we will have one of the three possibilities for the number of solutions.
1. No solution.
2. Exactly one solution.
3. Infinitely many solutions.

Before moving on to the next section we need to take a look at one more situation. The system of equations in (1) is called a nonhomogeneous system if at least one of the b_i's is not zero. If however all of the b_i's are zero we call the system homogeneous and the system will be,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned} \qquad (2)$$

Now, notice that in the homogeneous case we are guaranteed to have the following solution.

$$x_1 = x_2 = \cdots = x_n = 0$$

This solution is often called the **trivial solution**.

For homogeneous systems the fact above can be modified to the following.

Fact

Given a homogeneous system of equations, (2), we will have one of the two possibilities for the number of solutions.
1. Exactly one solution, the trivial solution
2. Infinitely many non-zero solutions in addition to the trivial solution.

In the second possibility we can say non-zero solution because if there are going to be infinitely many solutions and we know that one of them is the trivial solution then all the rest must have at least one of the x_i's be non-zero and hence we get a non-zero solution.

Review : Matrices and Vectors

This section is intended to be a catch all for many of the basic concepts that are used occasionally in working with systems of differential equations. There will not be a lot of details in this section, nor will we be working large numbers of examples. Also, in many cases we will not be looking at the general case since we won't need the general cases in our differential equations work.

Let's start with some of the basic notation for matrices. An $n \times m$ (this is often called the **size** or **dimension** of the matrix) matrix is a matrix with n rows and m columns and the entry in the ith row and j^{th} column is denoted by a_{ij}. A short hand method of writing a general $n \times m$ matrix is the following.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}_{n \times m} = \left(a_{ij} \right)_{n \times m}$$

The size or dimension of a matrix is subscripted as shown if required. If it's not required or clear from the problem the subscripted size is often dropped from the matrix.

Special Matrices
There are a few "special" matrices out there that we may use on occasion. The first special matrix is the **square matrix**. A square matrix is any matrix whose size (or dimension) is $n \times n$. In other words it has the same number of rows as columns. In a

square matrix the diagonal that starts in the upper left and ends in the lower right is often called the **main diagonal**.

The next two special matrices that we want to look at are the zero matrix and the identity matrix. The **zero matrix**, denoted $0_{n \, x \, m}$, is a matrix all of whose entries are zeroes. The **identity matrix** is a square n x n matrix, denoted I_n, whose main diagonals are all 1's and all the other elements are zero. Here are the general zero and identity matrices.

$$0_{n \, x \, m} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n \, x \, m} \qquad I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n \, x \, n}$$

In matrix arithmetic these two matrices will act in matrix work like zero and one act in the real number system.

The last two special matrices that we'll look at here are the **column matrix** and the **row matrix**. These are matrices that consist of a single column or a single row. In general they are,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \, x \, 1} \qquad\qquad y = \begin{pmatrix} y_1 & y_2 & \cdots & y_m \end{pmatrix}_{1 \, x \, m}$$

We will often refer to these as **vectors**.

Arithmetic

We next need to take a look at arithmetic involving matrices. We'll start with **addition** and **subtraction** of two matrices. So, suppose that we have two n x m matrices, A and B. The sum (or difference) of these two matrices is then,

$$A_{n \, x \, m} \pm B_{n \, x \, m} = \left(a_{ij} \right)_{n \, x \, m} \pm \left(b_{ij} \right)_{n \, x \, m} = \left(a_{ij} \pm b_{ij} \right)_{n \, x \, m}$$

The sum or difference of two matrices of the same size is a new matrix of identical size whose entries are the sum or difference of the corresponding entries from the original two matrices. Note that we can't add or subtract entries with different sizes.

Next, let's look at **scalar multiplication**. In scalar multiplication we are going to multiply a matrix A by a constant (sometimes called a scalar) α. In this case we get a new matrix whose entries have all been multiplied by the constant, α.

$$\alpha A_{n \, x \, m} = \alpha \left(a_{ij} \right)_{n \, x \, m} = \left(\alpha \, a_{ij} \right)_{n \, x \, m}$$

Example 1 Given the following two matrices,

$$A = \begin{pmatrix} 3 & -2 \\ -9 & 1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} -4 & 1 \\ 0 & -5 \end{pmatrix}$$

compute A-5B.

Solution
There isn't much to do here other than the work.

$$
\begin{aligned}
A - 5B &= \begin{pmatrix} 3 & -2 \\ -9 & 1 \end{pmatrix} - 5\begin{pmatrix} -4 & 1 \\ 0 & -5 \end{pmatrix} \\
&= \begin{pmatrix} 3 & -2 \\ -9 & 1 \end{pmatrix} - \begin{pmatrix} -20 & 5 \\ 0 & -25 \end{pmatrix} \\
&= \begin{pmatrix} 23 & -7 \\ -9 & 26 \end{pmatrix}
\end{aligned}
$$

We first multiplied all the entries of *B* by 5 then subtracted corresponding entries to get the entries in the new matrix.

The final matrix operation that we'll take a look at is **matrix multiplication**. Here we will start with two matrices, $A_{n \times p}$ and $B_{p \times m}$. Note that *A* must have the same number of columns as *B* has rows. If this isn't true then we can't perform the multiplication. If it is true then we can perform the following multiplication.

$$A_{n \times p} B_{p \times m} = \left(c_{ij}\right)_{n \times m}$$

The new matrix will have size *n* x *m* and the entry in the ith row and j^{th} column, c_{ij}, is found by multiplying row *i* of matrix *A* by column *j* of matrix *B*. This doesn't always make sense in words so let's look at an example.

Example 2 Given

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 6 & 1 \end{pmatrix}_{2 \times 3} \qquad\qquad B = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -4 & 3 & 1 & 0 \\ 0 & 3 & 0 & -2 \end{pmatrix}_{3 \times 4}$$

compute *AB*.

Solution
The new matrix will have size 2 x 4. The entry in row 1 and column 1 of the new matrix will be found by multiplying row 1 of *A* by column 1 of *B*. This means that we multiply corresponding entries from the row of *A* and the column of *B* and then add the results up. Here are a couple of the entries computed all the way out.

$$
\begin{aligned}
c_{11} &= (2)(1) + (-1)(-4) + (0)(0) = 6 \\
c_{13} &= (2)(-1) + (-1)(1) + (0)(0) = -3 \\
c_{24} &= (-3)(2) + (6)(0) + (1)(-2) = -8
\end{aligned}
$$

Here's the complete solution.
$$C = \begin{pmatrix} 6 & -3 & -3 & 4 \\ -27 & 21 & 9 & -8 \end{pmatrix}$$

In this last example notice that we could not have done the product *BA* since the number of columns of *B* does not match the number of row of *A*. It is important to note that just because we can compute *AB* doesn't mean that we can compute *BA*. Likewise, even if we can compute both *AB* and *BA* they may or may not be the same matrix.

Determinant

The next topic that we need to take a look at is the **determinant** of a matrix. The determinant is actually a function that takes a square matrix and converts it into a number. The actual formula for the function is somewhat complex and definitely beyond the scope of this review.

The main method for computing determinants of any square matrix is called the method of cofactors. Since we are going to be dealing almost exclusively with 2 x 2 matrices and the occasional 3 x 3 matrix we won't go into the method here. We can give simple formulas for each of these cases. The standard notation for the determinant of the matrix *A* is.

$$\det(A) = |A|$$

Here are the formulas for the determinant of 2 x 2 and 3 x 3 matrices.
$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 3 Find the determinant of each of the following matrices.
$$A = \begin{pmatrix} -9 & -18 \\ 2 & 4 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 2 & 3 & 1 \\ -1 & -6 & 7 \\ 4 & 5 & -1 \end{pmatrix}$$

Solution

For the 2 x 2 there isn't much to do other than to plug it into the formula.
$$\det(A) = \begin{vmatrix} -9 & -18 \\ 2 & 4 \end{vmatrix} = (-9)(4) - (-18)(2) = 0$$

For the 3 x 3 we could plug it into the formula, however unlike the 2 x 2 case this is not an easy formula to remember. There is an easier way to get the same result. A quicker way of getting the same result is to do the following. First write down the matrix and

tack a copy of the first two columns onto the end as follows.

$$\det(B) = \begin{vmatrix} 2 & 3 & 1 \\ -1 & -6 & 7 \\ 4 & 5 & -1 \end{vmatrix} \begin{matrix} 2 & 3 \\ -1 & -6 \\ 4 & 5 \end{matrix}$$

Now, notice that there are three diagonals that run from left to right and three diagonals that run from right to left. What we do is multiply the entries on each diagonal up and the if the diagonal runs from left to right we add them up and if the diagonal runs from right to left we subtract them.

Here is the work for this matrix.

$$\det(B) = \begin{vmatrix} 2 & 3 & 1 \\ -1 & -6 & 7 \\ 4 & 5 & -1 \end{vmatrix} \begin{matrix} 2 & 3 \\ -1 & -6 \\ 4 & 5 \end{matrix}$$

$$= (2)(-6)(-1) + (3)(7)(4) + (1)(-1)(5) -$$
$$(3)(-1)(-1) - (2)(7)(5) - (1)(-6)(4)$$

$$= 42$$

You can either use the formula or the short cut to get the determinant of a 3 x 3.

If the determinant of a matrix is zero we call that matrix **singular** and if the determinant of a matrix isn't zero we call the matrix **nonsingular**. The 2 x 2 matrix in the above example was singular while the 3 x 3 matrix is nonsingular.

Matrix Inverse

Next we need to take a look at the **inverse** of a matrix. Given a square matrix, A, of size nxn if we can find another matrix of the same size, B such that,

$$AB = BA = I_n$$

then we call B the **inverse** of A and denote it by $B=A^{-1}$.

Computing the inverse of a matrix, A, is fairly simple. First we form a new matrix,

$$(A \ I_n)$$

and then use the row operations from the previous [section](#) and try to convert this matrix into the form,

$$(I_n \ B)$$

If we can then B is the inverse of A. If we can't then there is no inverse of the matrix A.

Example 4 Find the inverse of the following matrix, if it exists.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -5 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

Solution
We first form the new matrix by tacking on the 3 x 3 identity matrix to this matrix. This is

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ -5 & -3 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

We will now use row operations to try and convert the first three columns to the 3 x 3 identity. In other words we want a 1 on the diagonal that starts at the upper left corner and zeroes in all the other entries in the first three columns.

If you think about it, this process is very similar to the process we used in the last section to solve systems, it just goes a little farther. Here is the work for this problem.

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ -5 & -3 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_1 \leftrightarrow R_3 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ -5 & -3 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{matrix} R_2 + 5R_1 \\ R_3 - 2R_1 \\ \Rightarrow \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & -5 & 0 & 1 & 5 \\ 0 & -1 & 3 & 1 & 0 & -2 \end{pmatrix} \begin{matrix} \frac{1}{2}R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & \frac{-5}{2} & 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & -1 & 3 & 1 & 0 & -2 \end{pmatrix} \begin{matrix} R_3 + R_2 \\ \Rightarrow \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & \frac{-5}{2} & 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{matrix} 2R_3 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & \frac{-5}{2} & 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix} \begin{matrix} R_2 + \frac{5}{2}R_3 \\ R_1 + R_3 \\ \Rightarrow \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 5 & 3 & 5 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 - R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 0 & 0 & -3 & -2 & -3 \\ 0 & 1 & 0 & 5 & 3 & 5 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$

So, we were able to convert the first three columns into the 3 x 3 identity matrix therefore the inverse exists and it is,

$$A^{-1} = \begin{pmatrix} -3 & -2 & -3 \\ 5 & 3 & 5 \\ 2 & 1 & 1 \end{pmatrix}$$

So, there was an example in which the inverse did exist. Let's take a look at an example in which the inverse doesn't exist.

Example 5 Find the inverse of the following matrix, provided it exists.
$$B = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$$
Solution
In this case we will tack on the 2 x 2 identity to get the new matrix and then try to convert

the first two columns to the 2 x 2 identity matrix.

$$\begin{pmatrix} 1 & -3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} \quad \begin{matrix} 2R_1 + R_2 \\ \Rightarrow \end{matrix} \quad \begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

And we don't need to go any farther. In order for the 2 x 2 identity to be in the first two columns we must have a 1 in the second entry of the second column and a 0 in the second entry of the first column. However, there is no way to get a 1 in the second entry of the second column that will keep a 0 in the second entry in the first column. Therefore, we can't get the 2 x 2 identity in the first two columns and hence the inverse of *B* doesn't exist.

We will leave off this discussion of inverses with the following fact.

Fact

Given a matrix *A*.
  1. If *A* is nonsingular then A^{-1} will exist.
  2. If *A* is singular then A^{-1} NOT will exist.

I'll leave it to you to verify this fact for the previous two examples.

Systems of Equations Revisited

We need to do a quick revisit of systems of equations. Let's start with a general system of equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

Now, covert each side into a vector to get,

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The left side of this equation can be thought of as a matrix multiplication.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Simplifying up the notation a little gives,

$$A\vec{x} = \vec{b} \tag{2}$$

where, $\vec{x}$ is a vector whose components are the unknowns in the original system of equations. We call (2) the matrix form of the system of equations (1) and solving (2) is equivalent to solving (1). The solving process is identical. The augmented matrix for (2) is

$$\left(A \ \vec{b} \right)$$

Once we have the augmented matrix we proceed as we did with a system that hasn't been wrote in matrix form.

We also have the following fact about solutions to (2).

Fact

Given the system of equation (2) we have one of the following three possibilities for solutions.
1. There will be no solutions.
2. There will be exactly one solution.
3. There will be infinitely many solutions.

In fact we can go a little farther now. Since we are assuming that we've got the same number of equations as unknowns the matrix A in (2) is a square matrix and so we can compute its determinant. This gives the following fact.

Fact

Given the system of equations in (2) we have the following.
1. If A is nonsingular then there will be exactly one solution to the system.
2. If A is singular then there will either be no solution or infinitely many solutions to the system.

The matrix form of a homogeneous system is

$$A\vec{x} = \vec{0} \tag{3}$$

where $\vec{0}$ is the vector of all zeroes. In the homogeneous system we are guaranteed to have a solution, $\vec{x} = \vec{0}$. The fact above for homogeneous systems is then,

Fact

Given the homogeneous system (3) we have the following.
1. If A is nonsingular then the only solution will be $\vec{x} = \vec{0}$.
2. If A is singular then there will be infinitely many nonzero solutions to the system.

Linear Independence/Linear Dependence
This is not the first time that we've seen this topic. We also saw linear independence and linear dependence back when we were looking at second order differential equations. In that section we were dealing with functions, but the concept is essentially the same here. If we start with n vectors,

$$\vec{x}_1, \ \vec{x}_2, \ \ldots, \ \vec{x}_n$$

If we can find constants, $c_1, c_2, \ldots, c_n$ with at least two nonzero such that

$$c_1 \vec{x}_1 + c_2 \vec{x}_2 + \ldots + c_n \vec{x}_n = \vec{0} \tag{4}$$

then we call the vectors linearly dependent. If the only constants that work in (4) are $c_1=0, c_2=0, \ldots, c_n=0$ then we call the vectors linearly independent.

If we further make the assumption that each of the n vectors has n components, *i.e.* each of the vectors look like,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

we can get a very simple test for linear independence and linear dependence. Note that this does not have to be the case, but in all of our work we will be working with n vectors each of which has n components.

Fact

Given the n vectors each with n components,

$$\vec{x}_1, \, \vec{x}_2, \, \ldots, \, \vec{x}_n$$

form the matrix,

$$X = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{pmatrix}$$

So, the matrix X is a matrix whose i^{th} column is the ith vector, $\vec{x}_i$. Then,

1. If X is nonsingular (*i.e.* det(X) is not zero) then the n vectors are linearly independent, and
2. if X is singular (*i.e.* det(X) = 0) then the n vectors are linearly dependent and the constants that make (4) true can be found by solving the system

$$X \vec{c} = \vec{0}$$

where $\vec{c}$ is a vector containing the constants in (4).

Example 6 Determine if the following set of vectors are linearly independent or linearly dependent. If they are linearly dependent find the relationship between them.

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{x}^{(2)} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}, \quad \vec{x}^{(3)} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}$$

Solution
So, the first thing to do is to form X and compute its determinant.

$$X = \begin{pmatrix} 1 & -2 & 6 \\ -3 & 1 & -2 \\ 5 & 4 & 1 \end{pmatrix} \quad \Rightarrow \quad \det(X) = -79$$

This matrix is non singular and so the vectors are linearly independent.

Example 7 Determine if the following set of vectors are linearly independent or linearly

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dependent. If they are linearly dependent find the relationship between them.

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \vec{x}^{(2)} = \begin{pmatrix} -4 \\ 1 \\ -6 \end{pmatrix}, \quad \vec{x}^{(3)} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

Solution
As with the last example first form X and compute its determinant.

$$X = \begin{pmatrix} 1 & -4 & 2 \\ -1 & 1 & -1 \\ 3 & -6 & 4 \end{pmatrix} \qquad \Rightarrow \qquad \det(X) = 0$$

So, these vectors are linearly dependent. We now need to find the relationship between the vectors. This means that we need to find constants that will make (4) true.

So we need to solve the system

$$X\,\vec{c} = \vec{0}$$

Here is the augmented matrix and the solution work for this system.

$$\begin{pmatrix} 1 & -4 & 2 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & -6 & 4 & 0 \end{pmatrix} \begin{matrix} R_2 + R_1 \\ R_3 - 3R_1 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & -4 & 2 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 6 & -2 & 0 \end{pmatrix} \begin{matrix} R_3 + 2R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & -4 & 2 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} -\frac{1}{3}R_2 \\ \Rightarrow \end{matrix}$$

$$\begin{pmatrix} 1 & -4 & 2 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 + 4R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} c_1 + \frac{2}{3}c_3 = 0 \\ c_2 - \frac{1}{3}c_3 = 0 \\ 0 = 0 \end{matrix} \Rightarrow \begin{matrix} c_1 = -\frac{2}{3}c_3 \\ c_2 = \frac{1}{3}c_3 \end{matrix}$$

Now, we would like actual values for the constants so, if use $c_3 = 3$ we get the following solution $c_1 = -2$, $c_2 = 1$, and $c_3 = 3$. The relationship is then.

$$-2\vec{x}^{(1)} + \vec{x}^{(2)} + 3\vec{x}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Calculus with Matrices
There really isn't a whole lot to this other than to just make sure that we can deal with calculus with matrices.

First, to this point we've only looked at matrices with numbers as entries, but the entries in a matrix can be functions as well. So we can look at matrices in the following form,

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

Now we can talk about differentiating and integrating a matrix of this form. To differentiate or integrate a matrix of this form all we do is differentiate or integrate the individual entries.

$$A'(t) = \begin{pmatrix} a'_{11}(t) & a'_{12}(t) & \cdots & a'_{1n}(t) \\ a'_{21}(t) & a'_{22}(t) & \cdots & a'_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a'_{m1}(t) & a'_{m2}(t) & \cdots & a'_{mn}(t) \end{pmatrix}$$

$$\int A(t)\,dt = \begin{pmatrix} \int a_{11}(t)\,dt & \int a_{12}(t)\,dt & \cdots & \int a_{1n}(t)\,dt \\ \int a_{21}(t)\,dt & \int a_{22}(t)\,dt & \cdots & \int a_{2n}(t)\,dt \\ \vdots & \vdots & & \vdots \\ \int a_{m1}(t)\,dt & \int a_{m2}(t)\,dt & \cdots & \int a_{mn}(t)\,dt \end{pmatrix}$$

So when we run across this kind of thing don't get excited about it. Just differentiate or integrate as we normally would.

In this section we saw a very condensed set of topics from linear algebra. When we get back to differential equations many of these topics will show up occasionally and you will at least need to know what the words mean.

The main topic from linear algebra that you must know however if you are going to be able to solve systems of differential equations is the topic of the next section.

Review : Eigenvalues and Eigenvectors

If you get nothing out of this quick review of linear algebra you must get this section. Without this section you will not be able to do any of the differential equations work that is in this chapter.

So let's start with the following. If we multiply an *n* x *n* matrix by an *n* x 1 vector we will get a new *n* x 1 vector back. In other words,

$$A\vec{\eta} = \vec{y}$$

What we want to know is if it is possible for the following to happen. Instead of just getting a brand new vector out of the multiplication is it possible instead to get the following,

$$A\vec{\eta} = \lambda\vec{\eta} \tag{1}$$

In other words is it possible, at least for certain λ and $\vec{\eta}$, to have matrix multiplication be the same as must multiplying the vector by a constant? Of course, we probably wouldn't

be talking about this if the answer was no. So, it is possible for this to happen, however, it won't happen for just any value of λ or $\vec{\eta}$. If we do happen to have a λ and $\vec{\eta}$ for which this works (and they will always come in pairs) then we call λ an **eigenvalue** of A and $\vec{\eta}$ an **eigenvector** of A.

So, how do we go about find the eigenvalues and eigenvectors for a matrix? Well first notice that that if $\vec{\eta} = \vec{0}$ then (1) is going to be true for any value of λ and so we are going to make the assumption that $\vec{\eta} \neq \vec{0}$. With that out of the way let's rewrite (1) a little.

$$A\vec{\eta} - \lambda\vec{\eta} = \vec{0}$$
$$A\vec{\eta} - \lambda I_n\vec{\eta} = \vec{0}$$
$$\left(A - \lambda I_n\right)\vec{\eta} = \vec{0}$$

Notice that before we factored out the $\vec{\eta}$ we added in the appropriately sized identity matrix. This is equivalent to multiplying things by a one and so doesn't change the value of anything. We needed to do this because with out it we would have had the difference of a matrix, A, and a constant, λ, and this can't be done. We now have the difference of two matrices of the same size which can be done.

So, with this rewrite we see that

$$\left(A - \lambda I_n\right)\vec{\eta} = \vec{0} \tag{2}$$

is equivalent to (1). In order to find the eigenvectors for a matrix we will need to solve a homogeneous system. Recall the fact from the previous section that we know that we will either have exactly one solution ($\vec{\eta} = \vec{0}$) or we will have infinitely many nonzero solutions. Since we've already said that don't want $\vec{\eta} = \vec{0}$ this means that we want the second case.

Knowing this will allow us to find the eigenvalues for a matrix. Recall from this fact that we will get the second case only if the matrix in the system is singular. Therefore we will need to determine the values of λ for which we get,

$$\det\left(A - \lambda I\right) = 0$$

Once we have the eigenvalues we can then go back and determine the eigenvectors for each eigenvalue. Let's take a look at a couple of quick facts about eigenvalues and eigenvectors.

Fact

If A is an n x n matrix then $\det\left(A - \lambda I\right) = 0$ is an nth degree polynomial. This polynomial is called the **characteristic polynomial**.

To find eigenvalues of a matrix all we need to do is solve a polynomial. That's generally not too bad provided we keep *n* small. Likewise this fact also tells us that for an *n* x *n* matrix, *A*, we will have *n* eigenvalues if we include all repeated eigenvalues.

Fact

If $\lambda_1, \lambda_2, ..., \lambda_n$ is the complete list of eigenvalues for *A* (including all repeated eigenvalues) then,

1. If λ occurs only once in the list then we call λ **simple**.
2. If λ occurs *k>1* times in the list then we say that λ has **multiplicity *k***.
3. If $\lambda_1, \lambda_2, ..., \lambda_k$ ($k \leq n$) are the simple eigenvalues in the list with corresponding eigenvectors $\vec{\eta}^{(1)}$, $\vec{\eta}^{(2)}$, ..., $\vec{\eta}^{(k)}$ then the eigenvectors are all linearly independent.
4. If λ is an eigenvalue of *k > 1* then λ will have anywhere from 1 to *k* linearly independent eigenvalues.

The usefulness of these facts will become apparent when we get back into differential equations since in that work we will want linearly independent solutions.

Let's work a couple of examples now to see how we actually go about finding eigenvalues and eigenvectors.

Example 1 Find the eigenvalues and eigenvectors of the following matrix.
$$A = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$$

Solution
The first thing that we need to do is find the eigenvalues. That means we need the following matrix,
$$A - \lambda I = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 7 \\ -1 & -6-\lambda \end{pmatrix}$$
In particular we need to determine where the determinant of this matrix is zero.
$$\det(A - \lambda I) = (2-\lambda)(-6-\lambda) + 7 = \lambda^2 + 4\lambda - 5 = (\lambda+5)(\lambda-1)$$

So, it looks like we will have two simple eigenvalues for this matrix, $\lambda_1 = -5$ and $\lambda_2 = 1$. We will now need to find the eigenvectors for each of these. Also note that according to the fact above, the two eigenvectors should be linearly independent.

To find the eigenvectors we simply plug in each eigenvalue into (2) and solve. So, let's do that.

$\lambda_1 = -5$:
In this case we need to solve the following system.

$$\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Recall that officially to solve this system we use the following augmented matrix.

$$\begin{pmatrix} 7 & 7 & 0 \\ -1 & -1 & 0 \end{pmatrix} \begin{array}{c} \frac{1}{7}R_1 + R_2 \\ \Rightarrow \end{array} \begin{pmatrix} 7 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Upon reducing down we see that we get a single equation

$$7\eta_1 + 7\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

that will yield an infinite number of solutions. This is expected behavior. Recall that we picked the eigenvalues so that the matrix would be singular and so we would get infinitely many solutions.

Notice as well that we could have identified this from the original system. This won't always be the case, but in the 2 x 2 case we can see from the system that one row will be a multiple of the other and so we will get infinite solutions. From this point on we won't be actually solving systems in these cases. We will just go straight to the equation and we can use either of the two rows for this equation.

Now, let's get back to the eigenvector, since that is what we were after. In general then the eigenvector will be any vector that satisfies the following,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \qquad ,\eta_2 \neq 0$$

To get this we used the solution to the equation that we found above.

We really don't want a general eigenvector however so we will pick a value for η_2 to get a specific eigenvector. We can choose anything (except $\eta_2=0$), so pick something that will make the eigenvector "nice". Note as well that since we've already assumed that the eigenvector is not zero we must choose a value that will not give us zero, which is why we want to avoid except $\eta_2=0$ in this case. Here's the eigenvector for this eigenvalue.

$$\vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \text{using } \eta_2 = 1$$

Now we get to do this all over again for the second eigenvalue.

$\lambda_2 = 1$:

We'll do much less work with this part than we did with the previous part. We will need to solve the following system.

$$\begin{pmatrix} 1 & 7 \\ -1 & -7 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Clearly both rows are multiples of each other and so we will get infinitely many solutions. We can choose to work with either row. We'll run with the first because to avoid having too many minus signs floating around. Doing this gives us,

$$\eta_1 + 7\eta_2 = 0 \qquad\qquad \eta_1 = -7\eta_2$$

Note that we can solve this for either of the two variables. However, with an eye towards working with these later on let's try to avoid as many fractions as possible. The eigenvector is then,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -7\eta_2 \\ \eta_2 \end{pmatrix} \qquad ,\eta_2 \neq 0$$

$$\vec{\eta}^{(2)} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}, \qquad\qquad \text{using } \eta_2 = 1$$

Summarizing we have,

$$\lambda_1 = -5 \qquad\qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 \qquad\qquad \vec{\eta}^{(1)} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$$

Note that the two eigenvectors are linearly independent as predicted.

Example 2 Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} 1 & -1 \\ \frac{4}{9} & -\frac{1}{3} \end{pmatrix}$$

Solution
This matrix has fractions in it. That's life so don't get excited about it. First we need the eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ \frac{4}{9} & -\frac{1}{3} - \lambda \end{vmatrix}$$

$$= (1-\lambda)\left(-\frac{1}{3} - \lambda\right) + \frac{4}{9}$$

$$= \lambda^2 - \frac{2}{3}\lambda + \frac{1}{9}$$

$$= \left(\lambda - \frac{1}{3}\right)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = \frac{1}{3}$$

So, it looks like we've got an eigenvalue of multiplicity 2 here. Remember that the power on the term will be the multiplicity.

Now, let's find the eigenvector(s). This one is going to be a little different from the first example. There is only one eigenvalue so let's do the work for that one. We will need to solve the following system,

$$\begin{pmatrix} \frac{2}{3} & -1 \\ \frac{4}{9} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad R_1 = \frac{3}{2} R_2$$

So, the rows are multiples of each other. We'll work with the first equation in this example to find the eigenvector.

$$\frac{2}{3}\eta_1 - \eta_2 = 0 \qquad\qquad \eta_2 = \frac{2}{3}\eta_1$$

Recall in the last example we decided that we wanted to make these as "nice" as possible and so should avoid fractions if we can. Some times, as in this case, we simply can't so we'll have to deal with it. In this case the eigenvector will be,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \frac{2}{3}\eta_1 \end{pmatrix}, \qquad\qquad \eta_1 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad\qquad \eta_1 = 3$$

Note that by careful choice of the variable in this case we were able to get rid of the fraction that we had. This is something that in general doesn't much matter if we do or not. However, when we get back to differential equations it will be easier on us if we don't have any fractions so we will usually try to eliminate them at this step.

Also in this case we are only going to get a single (linearly independent) eigenvector. We can get other eigenvectors, by choosing different values of η_1. However, each of these will be linearly dependent with the first eigenvector. If you're not convinced of this try it. Pick some values for η_1 and get a different vector and check to see if the two are linearly dependent.

Recall from the fact above that an eigenvalue of multiplicity k will have anywhere from 1 to k linearly independent eigenvectors. In this case we got one. For most of the 2 x 2 matrices that we'll be working with this will be the case, although it doesn't have to be. We can, on occasion, get two.

Example 3 Find the eigenvalues and eigenvectors of the following matrix.
$$A = \begin{pmatrix} -4 & -17 \\ 2 & 2 \end{pmatrix}$$

Solution
So, we'll start with the eigenvalues.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -4-\lambda & -17 \\ 2 & 2-\lambda \end{vmatrix} \\ &= (-4-\lambda)(2-\lambda) + 34 \\ &= \lambda^2 + 2\lambda + 26 \end{aligned}$$

This doesn't factor, so upon using the quadratic formula we arrive at,
$$\lambda_{1,2} = -1 \pm 5i$$

In this case we get complex eigenvalues which are definitely a fact of life with eigenvalue/eigenvector problems so get used to them.

Finding eigenvectors for complex eigenvalues is identical to the previous two examples, but it will be somewhat messier. So, let's do that.

$\lambda_1 = -1 + 5i$:
The system that we need to solve this time is
$$\begin{pmatrix} -4 - (-1 + 5i) & -17 \\ 2 & 2 - (-1 + 5i) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -3 - 5i & -17 \\ 2 & 3 - 5i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now, it's not super clear that the rows are multiples of each other, but they are. In this case we have,
$$R_1 = -\frac{1}{2}(3 + 5i) R_2$$

This is not something that you need to worry about, we just wanted to make the point. For the work that we'll be doing later on with differential equations we will just assume that we've done everything correctly and we've got two rows are multiples of each other. Therefore, all that we need to do here is pick one of the rows and work with it.

We'll work with the second row this time.
$$2\eta_1 + (3 - 5i)\eta_2 = 0$$

Now we can solve for either of the two variables. However, again looking forward to differential equations, we are going to need the "i" in the numerator so solve the equation in such a way as this will happen. Doing this gives,
$$2\eta_1 = -(3 - 5i)\eta_2$$
$$\eta_1 = -\frac{1}{2}(3 - 5i)\eta_2$$

So, the eigenvector in this case is
$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(3 - 5i)\eta_2 \\ \eta_2 \end{pmatrix}, \qquad \eta_2 \neq 0$$
$$\vec{\eta}^{(1)} = \begin{pmatrix} -3 + 5i \\ 2 \end{pmatrix}, \qquad \eta_2 = 2$$

As with the previous example we choose the value of the variable to clear out the

fraction.

Now, the work for the second eigenvector is almost identical and so we'll not dwell on that too much.

$\lambda_2 = -1-5i$:
The system that we need to solve here is

$$\begin{pmatrix} -4-(-1-5i) & -17 \\ 2 & 2-(-1-5i) \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3+5i & -17 \\ 2 & 3+5i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Working with the second row again gives,

$$2\eta_1 + (3+5i)\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\frac{1}{2}(3+5i)\eta_2$$

The eigenvector in this case is

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2}(3+5i)\eta_2 \\ \eta_2 \end{pmatrix}, \qquad \eta_2 \neq 0$$

$$\vec{\eta}^{(2)} = \begin{pmatrix} -3-5i \\ 2 \end{pmatrix}, \qquad \eta_2 = 2$$

Summarizing,

$$\lambda_1 = -1+5i \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -3+5i \\ 2 \end{pmatrix}$$

$$\lambda_1 = -1-5i \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -3-5i \\ 2 \end{pmatrix}$$

There is a nice fact that we can use to simplify the work when we get complex eigenvalues. We need a bit of terminology first however.

If we start with a complex number,

$$z = a+bi$$

then the **complex conjugate** of z is

$$\overline{z} = a-bi$$

To compute the complex conjugate of a complex number we simply change the sign on the term that contains the "i". The complex conjugate of a vector is just the conjugate of each of the vectors components.

We now have the following fact about complex eigenvalues and eigenvectors.

Fact

If A is an n x n matrix with only real numbers and if $\lambda_1 = a + bi$ is an eigenvalue with eigenvector $\vec{\eta}^{(1)}$. Then $\lambda_2 = a - bi$ is also an eigenvalue and it's eigenvector is the conjugate of $\vec{\eta}^{(1)}$.

This is a fact is something that you should feel free to use as you need to in our work.

Now, we need to work one final eigenvalue/eigenvector problem. To this point we've only worked with 2 x 2 matrices and we should work at least one that isn't 2 x 2. Also, we need to work one in which we get an eigenvalue of multiplicity greater than one that has more than one linearly independent eigenvector.

Example 4 Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution
Despite the fact that this is a 3 x 3 matrix, it still works the same as the 2 x 2 matrices that we've been working with. So, start with the eigenvalues

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 + 3\lambda + 2$$

$$= (\lambda - 2)(\lambda + 1)^2 \qquad \lambda_1 = 2, \ \lambda_{2,3} = -1$$

So, we've got a simple eigenvalue and an eigenvalue of multiplicity 2. Note that we used the same method of computing the determinant of a 3 x 3 matrix that we used in the previous section. We just didn't show the work.

Let's now get the eigenvectors. We'll start with the simple eigenvector.

$\lambda_1 = 2$:
Here we'll need to solve,

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This time, unlike the 2 x 2 cases we worked earlier, we actually need to solve the system. So let's do that.

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right) \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right) \begin{array}{c} R_2 + 2R_1 \\ R_3 - R_1 \\ \Rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array} \right)$$

$$-\tfrac{1}{3}R_2 \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \begin{matrix} R_3 - 3R_2 \\ R_1 + 2R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Going back to equations gives,

$$\eta_1 - \eta_3 = 0 \qquad \Rightarrow \qquad \eta_1 = \eta_3$$
$$\eta_2 - \eta_3 = 0 \qquad \Rightarrow \qquad \eta_2 = \eta_3$$

So, again we get infinitely many solutions as we should for eigenvectors. The eigenvector is then,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \eta_3 \\ \eta_3 \\ \eta_3 \end{pmatrix}, \qquad \eta_3 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \eta_3 = 1$$

Now, let's do the other eigenvalue.

$\lambda_2 = -1$:
Here we'll need to solve,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Okay, in this case is clear that all three rows are the same and so there isn't any reason to actually solve the system since we can clear out the bottom two rows to all zeroes in one step. The equation that we get then is,

$$\eta_1 + \eta_2 + \eta_3 = 0 \Rightarrow \qquad \eta_1 = -\eta_2 - \eta_3$$

So, in this case we get to pick two of the values for free and will still get infinitely many solutions. Here is the general eigenvector for this case,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} -\eta_2 - \eta_3 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \qquad \eta_2 \neq 0 \text{ and } \eta_3 \neq 0 \text{ at the same time}$$

Notice the restriction this time. Recall that we only require that the eigenvector not be the zero vector. This means that we can allow one or the other of the two variables to be zero, we just can't allow both of them to be zero at the same time!

What this means for us is that we are going to get two linearly independent eigenvectors this time. Here they are.

$$\vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \eta_2 = 0 \text{ and } \eta_3 = 1$$

$$\vec{\eta}^{(3)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \eta_2 = 1 \text{ and } \eta_3 = 0$$

Now when we talked about linear independent vectors in the last section we only looked at *n* vectors each with *n* components. We can still talk about linear independence in this case however. Recall back with we did linear independence for functions we saw at the time that if two functions were linearly dependent then they were multiples of each other. Well the same thing holds true for vectors. Two vectors will be linearly dependent if they are multiples of each other. In this case there is no way to get $\vec{\eta}^{(2)}$ by multiplying $\vec{\eta}^{(3)}$ by a constant. Therefore, these two vectors must be linearly independent.

So, summarizing up, here are the eigenvalues and eigenvectors for this matrix

$$\lambda_1 = 2 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -1 \qquad \vec{\eta}^{(3)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Systems of Differential Equations

In the introduction to this section we briefly discussed how a system of differential equations can arise from a population problem in which we keep track of the population of both the prey and the predator. It makes sense that the number of prey present will affect the number of the predator present. Likewise, the number of predator present will affect the number of prey present. Therefore the differential equation that governs the population of either the prey or the predator should in some way depend on the population of the other. This will lead to two differential equations that must be solved simultaneously in order to determine the population of the prey and the predator.

The whole point of this is to notice that systems of differential equations can arise quite easily from naturally occurring situations. Developing an effective predator-prey system

of differential equations is not the subject of this chapter. However, systems can arise from n^{th} order linear differential equations as well. Before we get into this however, let's write down a system and get some terminology out of the way.

We are going to be looking at first order, linear systems of differential equations. These terms mean the same thing that they have meant up to this point. The largest derivative anywhere in the system will be a first derivative and all unknown functions and their derivatives will only occur to the first power and will not be multiplied by other unknown functions. Here is an example of a system of first order, linear differential equations.

$$x_1' = x_1 + 2x_2$$
$$x_2' = 3x_1 + 2x_2$$

We call this kind of system a **coupled** system since knowledge of x_2 is required in order to find x_1 and likewise knowledge of x_1 is required to find x_2. We will worry about how to go about solving these later. At this point we are only interested in becoming familiar with some of the basics of systems.

Now, as mentioned earlier, we can write an nth order linear differential equation as a system. Let's see how that can be done.

Example 1 Write the following 2nd order differential equations as a system of first order, linear differential equations.
$$2y'' - 5y' + y = 0 \qquad y(3) = 6 \qquad y'(3) = -1$$

Solution
We can write higher order differential equations as a system with a very simple change of variable. We'll start by defining the following two new functions.
$$x_1(t) = y(t)$$
$$x_2(t) = y'(t)$$
Now notice that if we differentiate both sides of these we get,
$$x_1' = y' = x_2$$
$$x_2' = y'' = -\frac{1}{2}y + \frac{5}{2}y' = -\frac{1}{2}x_1 + \frac{5}{2}x_2$$
Note the use of the differential equation in the second equation. We can also convert the initial conditions over to the new functions.
$$x_1(3) = y(3) = 6$$
$$x_2(3) = y'(3) = -1$$

Putting all of this together gives the following system of differential equations.
$$x_1' = x_2 \qquad\qquad x_1(3) = 6$$
$$x_2' = -\frac{1}{2}x_1 + \frac{5}{2}x_2 \qquad x_2(3) = -1$$

We will call the system in the above example an **Initial Value Problem** just as we did for differential equations with initial conditions.

Let's take a look at another example.

Example 2 Write the following differential equations as a system of first order, linear differential equations.
$$y^{(4)} + 3y'' - \sin(t)\,y' + 8y = t^2 \qquad y(0) = 1 \quad y'(0) = 2 \quad y''(0) = 3 \quad y'''(0) = 4$$

Solution
Just as we did in the last example we'll need to define some new functions. This time we'll need 4 new functions.

$$x_1 = y \qquad \Rightarrow \qquad x_1' = y' = x_2$$
$$x_2 = y' \qquad \Rightarrow \qquad x_2' = y'' = x_3$$
$$x_3 = y'' \qquad \Rightarrow \qquad x_3' = y''' = x_4$$
$$x_4 = y''' \qquad \Rightarrow \qquad x_4' = y^{(4)} = -8y + \sin(t)\,y' - 3y'' = -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2$$

The system along with the initial conditions is then,

$$x_1' = x_2 \qquad\qquad\qquad\qquad x_1(0) = 1$$
$$x_2' = x_3 \qquad\qquad\qquad\qquad x_2(0) = 2$$
$$x_3' = x_4 \qquad\qquad\qquad\qquad x_3(0) = 3$$
$$x_4' = -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 \qquad x_4(0) = 4$$

Now, when we finally get around to solving these we will see that we generally don't solve systems in the form that we've given them in this section. Systems of differential equations can be converted to **matrix form** and this is the form that we usually use in solving systems.

Example 3 Convert the system the following system to matrix from.
$$x_1' = 4x_1 + 7x_2$$
$$x_2' = -2x_1 - 5x_2$$

Solution
First write the system so that each side is a vector.
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4x_1 + 7x_2 \\ -2x_1 - 5x_2 \end{pmatrix}$$

Now the right side can be written as a matrix multiplication,
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Now, if we define,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

then,

$$\vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

The system can then be wrote in the matrix form,

$$\vec{x}' = \begin{pmatrix} 4 & 7 \\ -2 & -5 \end{pmatrix} \vec{x}$$

Example 4 Convert the systems from Examples 1 and 2 into matrix form.

Solution
We'll start with the system from Example 1.

$$x_1' = x_2 \qquad\qquad x_1(3) = 6$$

$$x_2' = -\frac{1}{2}x_1 + \frac{5}{2}x_2 \qquad x_2(3) = -1$$

First define,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The system is then,

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix} \vec{x} \qquad \vec{x}(3) = \begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

Now, let's do the system from Example 2.

$$x_1' = x_2 \qquad\qquad\qquad x_1(0) = 1$$

$$x_2' = x_3 \qquad\qquad\qquad x_2(0) = 2$$

$$x_3' = x_4 \qquad\qquad\qquad x_3(0) = 3$$

$$x_4' = -8x_1 + \sin(t)x_2 - 3x_3 + t^2 \qquad x_4(0) = 4$$

In this case we need to be careful with the t^2 in the last equation. We'll start by writing the system as a vector again and then break it up into two vectors, one vector that contains the unknown functions and the other that contains any known functions.

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -8x_1 + \sin(t)x_2 - 3x_3 + t^2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -8x_1 + \sin(t)x_2 - 3x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix}$$

Now, the first vector can now be written as a matrix multiplication and we'll leave the

second vector alone.
$$\vec{x}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & \sin(t) & -3 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

where,
$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}$$

Note that occasionally for "large" systems such as this we will one step farther and write the system as,
$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

The last thing that we need to do in this section is get a bit of terminology out of the way. Starting with
$$\vec{x}' = A\vec{x} + \vec{g}(t)$$
we say that the system is **homogeneous** if $\vec{g}(t) = \vec{0}$ and we say the system is

nonhomogeneous if $\vec{g}(t) \neq \vec{0}$.

Solutions to Systems

Now that we've got some of the basic out of the way for systems of differential equations it's time to start thinking about how to solve a system of differential equations. We will start with the homogeneous system written in matrix form,
$$\vec{x}' = A\vec{x} \qquad \text{(1)}$$
where, A is an n x n matrix and $\vec{x}$ is a vector whose components are the unknown functions in the system.

Now, if we start with $n = 1$ then the system reduces to a fairly simple linear (or separable) first order differential equation.
$$x' = ax$$
and this has the following solution,
$$x(t) = c\mathbf{e}^{at}$$

So, let's use this as a guide and for a general n let's see if
$$\vec{x}(t) = \vec{\eta}\,\mathbf{e}^{rt} \qquad \text{(2)}$$

will be a solution. Note that the only real difference here is that we let the constant in front of the exponential be a vector. All we need to do then is plug this into the differential equation and see what we get. First notice that the derivative is,

$$\vec{x}'(t) = r\vec{\eta}\,\mathbf{e}^{rt}$$

So upon plugging the guess into the differential equation we get,

$$r\vec{\eta}\mathbf{e}^{rt} = A\vec{\eta}\mathbf{e}^{rt}$$

$$\left(A\vec{\eta} - r\vec{\eta}\right)\mathbf{e}^{rt} = \vec{0}$$

$$\left(A - rI\right)\vec{\eta}\mathbf{e}^{rt} = \vec{0}$$

Now, since we know that exponentials are not zero we can drop that portion and we then see that in order for (2) to be a solution to (1) then we must have

$$\left(A - rI\right)\vec{\eta} = \vec{0}$$

Or, in order for (2) to be a solution to (1), r and $\vec{\eta}$ must be an eigenvalue and eigenvector for the matrix A.

Therefore, in order to solve (1) we first find the eigenvalues and eigenvectors of the matrix A and then we can form solutions using (2). There are going to be three cases that we'll need to look at. The cases are real, distinct eigenvalues, complex eigenvalues and repeated eigenvalues.

None of this tells us how to completely solve a system of differential equations. We'll need the following couple of facts to do this.

Fact

1. If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two solutions to a homogeneous system, (1), then

    $$c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

    is also a solution to the system.
2. Suppose that A is an n x n matrix and suppose that $\vec{x}_1(t)$, $\vec{x}_2(t)$, …, $\vec{x}_n(t)$ are solutions to a homogeneous system, (1). Define,

    $$X = \left(\vec{x}_1 \ \vec{x}_2 \ \cdots \ \vec{x}_n\right)$$

    In other words, X is a matrix whose ith column is the ith solution. Now define,

    $$W = \det\left(X\right)$$

    We call W the **Wronskian**. If $W \neq 0$ then the solutions form a **fundamental set of solutions** and the general solution to the system is,

    $$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

Note that if we have a fundamental set of solutions then the solutions are also going to be linearly independent. Likewise, if we have a set of linearly independent solutions then they will also be a fundamental set of solutions since the Wronskian will not be zero.

Phase Plane

Before proceeding with actually solving systems of differential equations there's one topic that we need to take a look at. This is a topic that's not always taught in a differential equations class but in case you're in a course where it is taught we should cover it so that you are prepared for it.

Let's start with a general homogeneous system,
$$\vec{x}' = A\vec{x} \tag{1}$$

Notice that
$$\vec{x} = \vec{0}$$
is a solution to the system of differential equations. What we'd like to ask is, do the other solutions to the system approach this solution as t increases or do they move away from this solution? We did something similar to this when we classified equilibrium solutions in a previous section. In fact, what we're doing here is simply an extension of this idea to systems of differential equations.

The solution $\vec{x} = \vec{0}$ is called an **equilibrium solution** for the system. As with the single differential equations case, equilibrium solutions are those solutions for which
$$A\vec{x} = \vec{0}$$

We are going to assume that A is a nonsingular matrix and hence will have only one solution,
$$\vec{x} = \vec{0}$$
and so we will have only one equilibrium solution.

Back in the single differential equation case recall that we started by choosing values of y and plugging these into the function $f(y)$ to determine values of y'. We then used these values to sketch tangents to the solution at that particular value of y. From this we could sketch in some solutions and use this information to classify the equilibrium solutions.

We are going to do something similar here, but it will be slightly different as well. First we are going to restrict ourselves down to the 2 x 2 case. So, we'll be looking at systems of the form,
$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{x}$$

Solutions to this system will be of the form,
$$\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

and our single equilibrium solution will be,

$$\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In the single differential equation case we were able to sketch the solution, *y(t)* in the *y-t* plane and see actual solutions. However, this would somewhat difficult in this case since our solutions are actually vectors. What we're going to do here is think of the solutions to the system as points in the x_1-x_2 plane and plot these points. Our equilibrium solution will correspond to the origin of x_1-x_2 plane and the x_1-x_2 plane is called the **phase plane**.

To sketch a solution in the phase plane we can pick values of *t* and plug these into the solution. This gives us a point in the x_1-x_2 or phase plane that we can plot. Doing this for many values of *t* will then give us a sketch of what the solution will be doing in the phase plane. A sketch of a particular solution in the phase plane is called the **trajectory** of the solution. Once we have the trajectory of a solution sketched we can then ask whether or not the solution will approach the equilibrium solution as *t* increases.

We would like to be able to sketch trajectories without actually having solutions in hand. There are a couple of ways to do this. We'll look at one of those here and we'll look at the other in the next couple of sections.

One way to get a sketch of trajectories is to do something similar to what we did the first time we looked at equilibrium solutions. We can choose values of $\vec{x}$ (note that these will be points in the phase plane) and compute $A\vec{x}$. This will give a vector that represents $\vec{x}'$ at that particular solution. As with the single differential equation case this vector will be tangent to the trajectory at that point. We can sketch a bunch of the tangent vectors and then sketch in the trajectories.

This is a fairly work intensive way of doing these and isn't the way to do them in general. However, it is a way to get trajectories without doing any solution work. All we need is the system of differential equations. Let's take a quick look at an example.

Example 1 Sketch some trajectories for the system,

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 3x_1 + 2x_2 \end{aligned} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}$$

Solution
So, what we need to do is pick some points in the phase plane, plug them into the right side of the system. We'll do this for a couple of points.

$$\vec{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -7 \\ -13 \end{pmatrix}$$


So, what does this tell us? Well at the point (-1, 1) in the phase plane there will be a vector pointing in the direction $\langle 1, -1 \rangle$. At the point (2,0) there will be a vector pointing in the direction $\langle 2, 6 \rangle$. At the point (-3,-2) there will be a vector pointing in the direction $\langle -7, -13 \rangle$.

Doing this for a large number of points in the phase plane will give the following sketch of vectors.
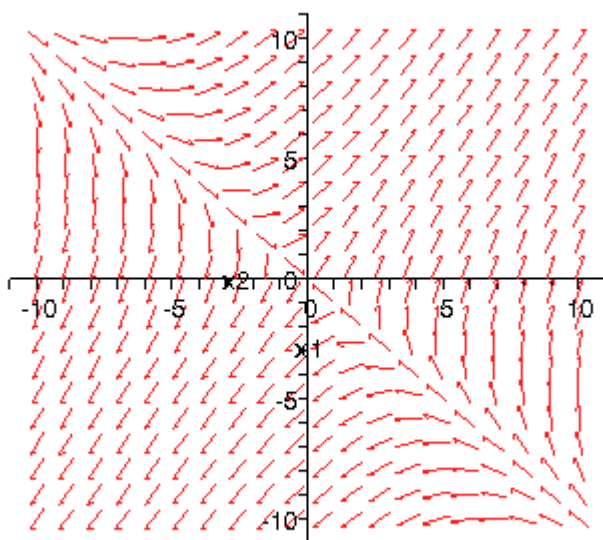
Now all we need to do is sketch in some trajectories. To do this all we need to do is remember that the vectors in the sketch above are tangent to the trajectories. Also the direction of the vectors give the direction of the trajectory as *t* increases so we can show the time dependence of the solution by adding in arrows to the trajectories.

Doing this gives the following sketch.

http://tutorial.math.lamar.edu/terms.asp
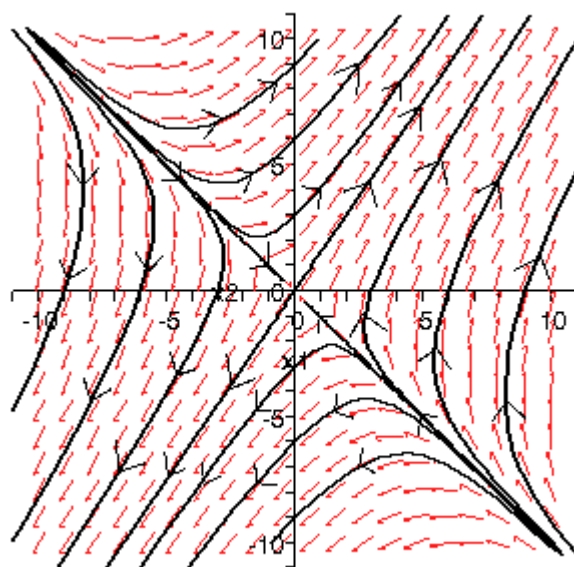
This sketch is called the **phase portrait**. Usually phase portraits only include the trajectories of the solutions and not any vectors. All of our phase portraits form this point on will only include the trajectories.

In this case it looks like most of the solutions will start away from the equilibrium solution then as *t* starts to increase they move in towards the equilibrium solution and then eventually start moving away from the equilibrium solution again.

There seem to be four solutions that have slightly different behaviors. It looks like two of the solutions will start at (or near at least) the equilibrium solution and them move straight away from it while two other solution start away from the equilibrium solution and then move straight in towards the equilibrium solution.

In these kinds of cases we call the equilibrium point a **saddle point** and we call the equilibrium point in this case **unstable** since all but two of the solutions are moving away from it as *t* increases.

As we noted earlier this is not generally the way that we will sketch trajectories. All we really need to get the trajectories are the eigenvalues and eigenvectors of the matrix *A*. We will see how to do this over the next couple of sections as we solve the systems.

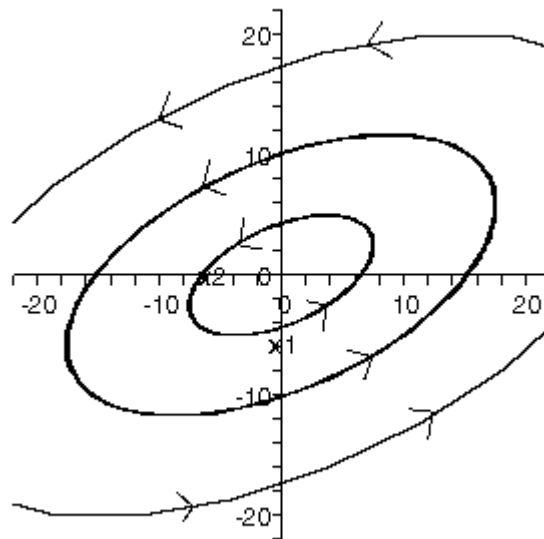Here are a few more phase portraits so you can see some more possible phase portraits.

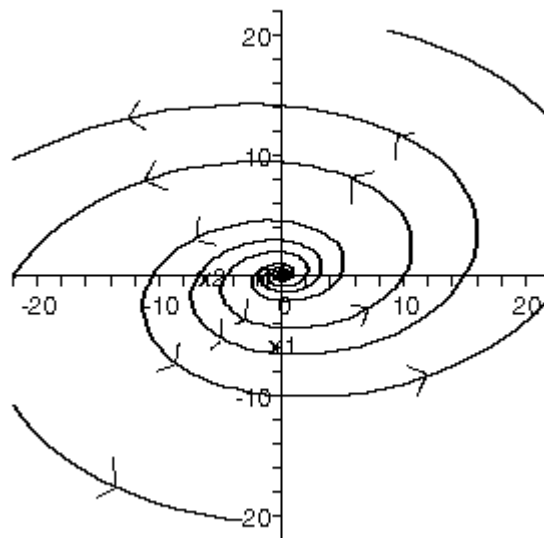## Node - Asymptotically Stable
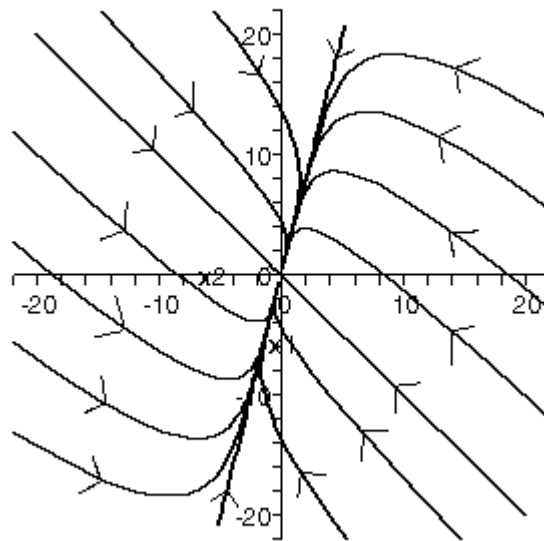
Improper Node - Unstable
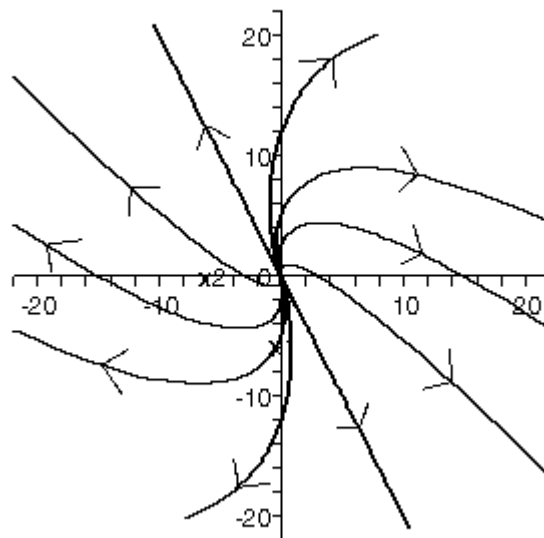
Center - Stable

Sprial - Unstable

Not all possible phase portraits have been shown here. These are here to show you some of the possibilities. Note as well that with the two nodes and the spiral the direction of motion could just as easily be reversed which would also switch the stability of those phase portraits.

Also notice the difference between **stable** and **asymptotically stable**. An asymptotically stable node or spiral will move in towards the equilibrium point as *t* increases whereas, a

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center (which is always stable) will just move around the equilibrium point but never actually move in towards it.

We will solve the systems that gave the phase portraits above in the next few sections.

Real, Distinct Eigenvalues

It's now time to start solving systems of differential equations. We've seen that solutions to the system,

$$\vec{x}' = A\vec{x}$$

will be of the form

$$\vec{x} = \vec{\eta}\mathbf{e}^{\lambda t}$$

where λ and $\vec{\eta}$ are eigenvalues and eigenvectors of the matrix A. We will be working with 2 x 2 systems so this means that we are going to be looking for two solutions, $\vec{x}_1(t)$ and $\vec{x}_2(t)$, where the determinant of the matrix,

$$X = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 \end{pmatrix}$$

is nonzero.

We are going to start by looking at the case where our two eigenvalues, λ_1 and λ_2 are real and distinct. In other words they will be real, simple eigenvalues. Recall as well that the eigenvectors for simple eigenvalues are linearly independent. This means that the solutions we get from these will also be linearly independent. If the solutions are linearly independent the matrix X must be nonsingular and hence these two solutions will be a fundamental set of solutions. The general solution in this case will then be,

$$\vec{x}(t) = c_1 \mathbf{e}^{\lambda_1 t}\vec{\eta}^{(1)} + c_2 \mathbf{e}^{\lambda_2 t}\vec{\eta}^{(2)}$$

Note that each of our examples will actually be broken into two examples. The first example will be solving the system and the second example will be sketching the phase portrait for the system. Phase portraits are not always taught in a differential equations course and so we'll strip those out of the solution process so that if you haven't covered them in your class you can ignore the phase portrait example for the system.

Example 1 Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

Solution
So, the first thing that we need to do is find the eigenvalues for the matrix.

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} \\
&= \lambda^2 - 3\lambda - 4 \\
&= (\lambda + 1)(\lambda - 4) \qquad \Rightarrow \qquad \lambda_1 = -1, \ \lambda_2 = 4
\end{aligned}$$

Now let's find the eigenvectors for each of these.

$\lambda_1 = -1$:
We'll need to solve,

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 2\eta_1 + 2\eta_2 = 0 \quad \Rightarrow \quad \eta_1 = -\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \quad \Rightarrow \quad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \eta_2 = 1$$

$\lambda_2 = 4$:
We'll need to solve,

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -3\eta_1 + 2\eta_2 = 0 \quad \Rightarrow \quad \eta_1 = \frac{2}{3}\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \frac{2}{3}\eta_2 \\ \eta_2 \end{pmatrix} \quad \Rightarrow \quad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \eta_2 = 3$$

Then general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, we need to find the constants. To do this we simply need to apply the initial conditions.

$$\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

All we need to do now is multiply the constants through and we then get two equations (one for each row) that we can solve for the constants. This gives,

$$\left.\begin{array}{r} -c_1 + 2c_2 = 0 \\ c_1 + 3c_2 = -4 \end{array}\right\} \quad \Rightarrow \quad c_1 = -\frac{8}{5}, \ c_2 = -\frac{4}{5}$$

The solution is then,

$$\vec{x}(t) = -\frac{8}{5}\mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5}\mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, let's take a look at the phase portrait for the system.

Example 2 Sketch the phase portrait for the following system.

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x}$$

Solution
From the last example we know that the eigenvalues and eigenvectors for this system are,

$$\lambda_1 = -1 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4 \qquad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

It turns out that this is all the information that we will need to sketch the direction field. We will relate things back to our solution however so that we can see that things are going correctly.

Well start by sketching lines that follow the direction of the two eigenvectors. This gives,
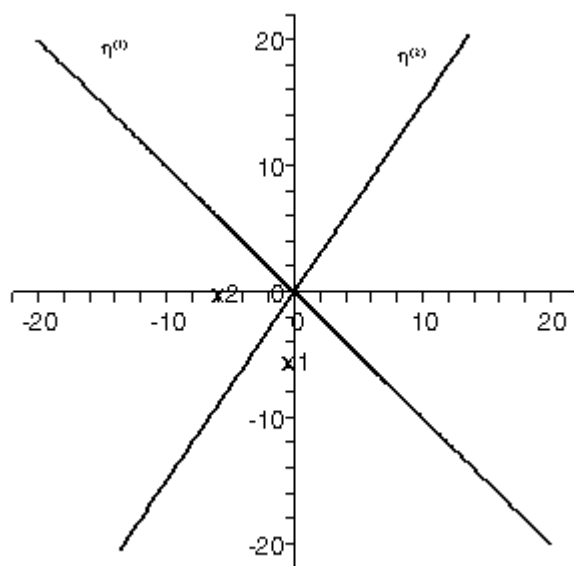

Now, from the first example our general solution is

$$\vec{x}(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

If we have $c_2 = 0$ then the solution is an exponential times a vector and that all the exponential does is affect the magnitude of the vector and the constant c_1 will affect both the sign and the magnitude of the vector. In other words, the trajectory in this case will be a straight line that is parallel to the vector, $\vec{\eta}^{(1)}$. Also notice that as t increases the exponential will get smaller and smaller and hence the trajectory will be moving in towards the origin. If $c_1 > 0$ the trajectory will be in Quadrant II and if $c_1 < 0$ the trajectory will be in Quadrant IV.

So the line in the graph above marked with $\vec{\eta}^{(1)}$ will be a sketch of the trajectory corresponding to $c_2 = 0$ and this trajectory will approach the origin as t increases.

If we now turn things around and look at the solution corresponding to have $c_1 = 0$ we will have a trajectory that is parallel to $\vec{\eta}^{(2)}$. Also, since the exponential will increase as t increases and so in this case the trajectory will now move away from the origin as t increases. We will denote this with arrows on the lines in the graph above.
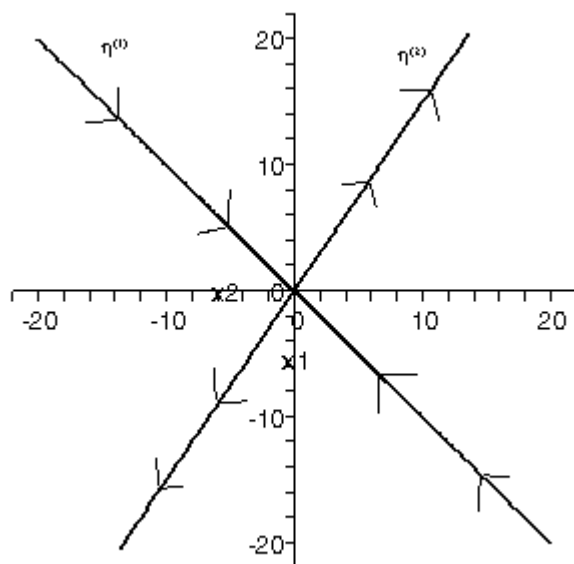

Notice that we could have gotten this information with actually going to the solution. All we really need to do is look at the eigenvalues. Eigenvalues that are negative will correspond to solutions that will move towards the origin as t increase in a direction that is parallel to its eigenvector. Likewise, eigenvalues that are positive move away from the origin as t increases in a direction that will be parallel to its eigenvector.

If both constants are in the solution we will have a combination of these behaviors. For large negative t's the solution will be dominated by the portion that has the negative eigenvalue since in these cases the exponent will be large and positive. Trajectories for large negative t's will be parallel to $\vec{\eta}^{(1)}$ and moving in the same direction.

Solutions for large positive t's will be dominated by the portion with the positive eigenvalue. Trajectories in this case will be parallel to $\vec{\eta}^{(2)}$ and moving in the same direction.

In general, it looks like trajectories will start "near" $\vec{\eta}^{(1)}$, move in towards the origin and then as they get closer to the origin they will start moving towards $\vec{\eta}^{(2)}$ and then continue

up along this vector. Sketching some of these in will give the following phase portrait. Here is a sketch of this with the trajectories corresponding to the eigenvectors marked in blue.
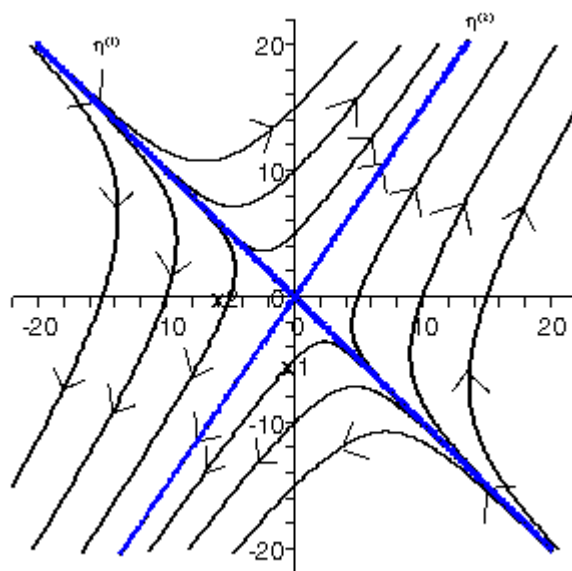
In this case the equilibrium solution (0,0) is called a **saddle point** and is unstable. In this case unstable means that solution move away from it as *t* increases.

So, we've solved a system in matrix form, but remember that we started out without the systems in matrix form. Now let's take a quick look at an example of a system that isn't in matrix form initially.

Example 3 Find the solution to the following system.
$$x_1' = x_1 + 2x_2 \qquad\qquad x_1(0) = 0$$
$$x_2' = 3x_1 + 2x_2 \qquad\qquad x_2(0) = -4$$

Solution
We first need to convert this into matrix form. This is easy enough. Here is the matrix form of the system.
$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}, \qquad\qquad \vec{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

This is just the system from the first example and so we've already got the solution to this system. Here it is.
$$\vec{x}(t) = -\frac{8}{5}\mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5}\mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, since we want to solution to the system not in matrix form let's go one step farther here. Let's multiply the constants and exponentials into the vectors and then add up the two vectors.

$$\vec{x}(t) = \begin{pmatrix} \frac{8}{5}\mathbf{e}^{-t} \\ -\frac{8}{5}\mathbf{e}^{-t} \end{pmatrix} - \begin{pmatrix} \frac{8}{5}\mathbf{e}^{4t} \\ \frac{12}{5}\mathbf{e}^{4t} \end{pmatrix} = \begin{pmatrix} \frac{8}{5}\mathbf{e}^{-t} - \frac{8}{5}\mathbf{e}^{4t} \\ -\frac{8}{5}\mathbf{e}^{-t} - \frac{12}{5}\mathbf{e}^{4t} \end{pmatrix}$$

Now, recall,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

So, the solution to the system is then,

$$x_1(t) = \frac{8}{5}\mathbf{e}^{-t} - \frac{8}{5}\mathbf{e}^{4t}$$

$$x_2(t) = -\frac{8}{5}\mathbf{e}^{-t} - \frac{12}{5}\mathbf{e}^{4t}$$

Let's work another example.

Example 4 Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution
So, the first thing that we need to do is find the eigenvalues for the matrix.

$$\det(A - \lambda I) = \begin{vmatrix} -5 - \lambda & 1 \\ 4 & -2 - \lambda \end{vmatrix}$$

$$= \lambda^2 + 7\lambda + 6$$

$$= (\lambda + 1)(\lambda + 6) \qquad \Rightarrow \qquad \lambda_1 = -1, \ \lambda_2 = -6$$

Now let's find the eigenvectors for each of these.

$\lambda_1 = -1$:
We'll need to solve,

$$\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -4\eta_1 + \eta_2 = 0 \quad \Rightarrow \quad \eta_2 = 4\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ 4\eta_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \eta_1 = 1$$

$\lambda_2 = -6$:
We'll need to solve,

$$\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \eta_1 + \eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \eta_2 = 1$$

Then general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 \mathbf{e}^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now, we need to find the constants. To do this we simply need to apply the initial conditions.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now solve the system for the constants.

$$\left. \begin{array}{c} c_1 - c_2 = 1 \\ 4c_1 + c_2 = 2 \end{array} \right\} \qquad \Rightarrow \qquad c_1 = \frac{3}{5}, \; c_2 = -\frac{2}{5}$$

The solution is then,

$$\vec{x}(t) = \frac{3}{5} \mathbf{e}^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \frac{2}{5} \mathbf{e}^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now let's find the phase portrait for this system.

Example 5 Sketch the phase portrait for the following system.

$$\vec{x}' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$$

Solution
From the last example we know that the eigenvalues and eigenvectors for this system are,

$$\lambda_1 = -1 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\lambda_2 = -6 \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

This one is a little different from the first one. However it starts in the same way. We'll first sketch the trajectories corresponding to the eigenvectors. Notice as well that both of the eigenvalues are negative and so trajectories for these will move in towards the origin as *t* increases. When we sketch the trajectories we'll add in arrows to denote the direction they take as *t* increases. Here is the sketch of these trajectories.
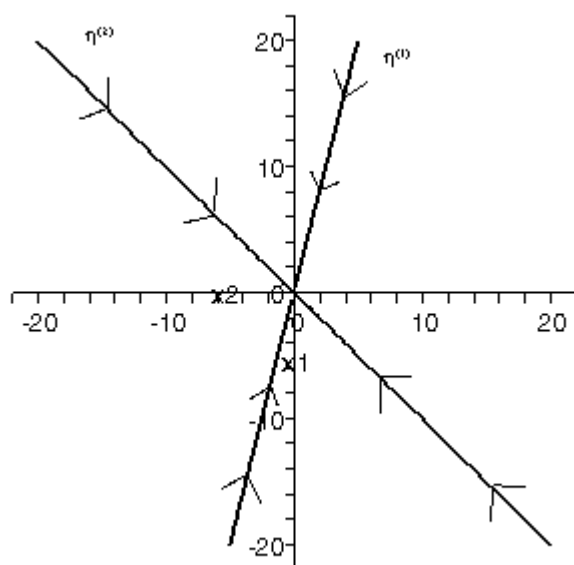
Now, here is where the slight difference from the first phase portrait comes up. All of the trajectories will move in towards the origin as *t* increases since both of the eigenvalues are negative. The issue that we need to decide upon is just how they do this. This is actually easier than it might appear to be at first.

The second eigenvalue is larger than the first. For large and positive *t*'s this means that the solution for this eigenvalue will be smaller than the solution for the first eigenvalue. Therefore, as *t* increases the trajectory will move in towards the origin and do so parallel to $\vec{\eta}^{(1)}$. Likewise, since the second eigenvalue is larger than the first this solution will dominate for large and negative *t*'s. Therefore, as we decrease *t* the trajectory will move away from the origin and do so parallel to $\vec{\eta}^{(2)}$.

Adding in some trajectories gives the following sketch.
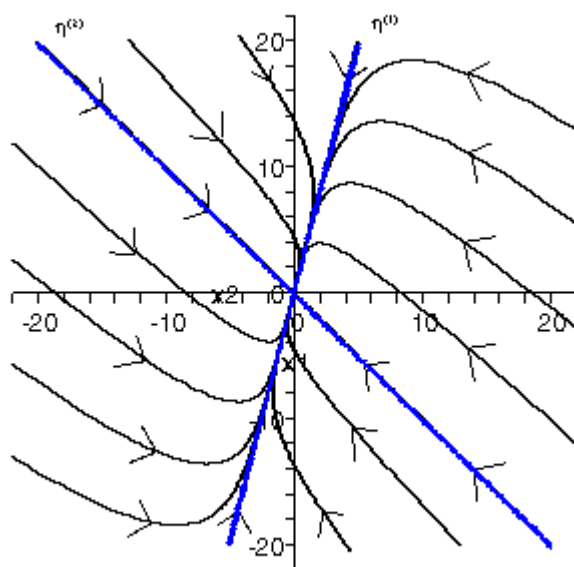
In these cases we call the equilibrium solution (0,0) a **node** and it is asymptotically stable. Equilibrium solutions are asymptotically stable if all the trajectories move in towards it as *t* increases.

Note that nodes can also be unstable. In the last example if both of the eigenvalues had been positive all the trajectories would have moved away from the origin and in this case the equilibrium solution would have been unstable.

Before moving on to the next section we need to do one more example. When we first started talking about systems it was mentioned that we can convert a higher order differential equation into a system. We need to do an example like this so we can see how to solve higher order differential equations using systems.

Example 6 Convert the following differential equation into a system, solve the system and use this solution to get the solution to the original differential equation.
$$2y'' + 5y' - 3y = 0, \qquad y(0) = -4 \quad y'(0) = 9$$

Solution
So, we first need to convert this into a system. Here's the change of variables,

$$x_1 = y \qquad x_1' = y' = x_2$$

$$x_2 = y' \qquad x_2' = y'' = \frac{3}{2}y - \frac{5}{2}y' = \frac{3}{2}x_1 - \frac{5}{2}x_2$$

The system is then,

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ \frac{3}{2} & -\frac{5}{2} \end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} -4 \\ 9 \end{pmatrix}$$

where,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

Now we need to find the eigenvalues for the matrix.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ \frac{3}{2} & -\frac{5}{2} - \lambda \end{vmatrix}$$

$$= \lambda^2 + \tfrac{5}{2}\lambda - \tfrac{3}{2}$$

$$= \tfrac{1}{2}(\lambda + 3)(2\lambda - 1) \qquad\qquad \lambda_1 = -3, \ \lambda_2 = \frac{1}{2}$$

Now let's find the eigenvectors.

$\lambda_1 = -3$:
We'll need to solve,

$$\begin{pmatrix} 3 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad 3\eta_1 + \eta_2 = 0 \qquad \Rightarrow \qquad \eta_2 = -3\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -3\eta_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \eta_1 = 1$$

$\lambda_2 = \frac{1}{2}$:
We'll need to solve,

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \qquad -\frac{1}{2}\eta_1 + \eta_2 = 0 \quad \Rightarrow \quad \eta_2 = \frac{1}{2}\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \frac{1}{2}\eta_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \eta_1 = 2$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \mathbf{e}^{\frac{t}{2}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Apply the initial condition.

$$\begin{pmatrix} -4 \\ 9 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

This gives the system of equations that we can solve for the constants.

$$\left.\begin{array}{c} c_1 + 2c_2 = -4 \\ -3c_1 + c_2 = 9 \end{array}\right\} \qquad \Rightarrow \qquad c_1 = -\frac{22}{7}, \ c_2 = -\frac{3}{7}$$

The actual solution to the system is then,

$$\vec{x}(t) = -\frac{22}{7}\mathbf{e}^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} - \frac{3}{7}\mathbf{e}^{\frac{t}{2}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Now recalling that,

$$\vec{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

we can see that the solution to the original differential equation is just the top row of the solution to the matrix system. The solution to the original differential equation is then,

$$y(t) = -\frac{22}{7}\mathbf{e}^{-3t} - \frac{6}{7}\mathbf{e}^{\frac{t}{2}}$$

Notice that as a check, in this case, the bottom row should be the derivative of the top row.

Complex Eigenvalues

In this section we will look at solutions to

$$\vec{x}' = A\vec{x}$$

where the eigenvalues of the matrix *A* are complex. With complex eigenvalues we are going to have the same problem that we had back when we were looking at second order differential equations. We want our solutions to only have real numbers in them, however since our solutions to systems are of the form,

$$\vec{x} = \vec{\eta}\mathbf{e}^{\lambda t}$$

we are going to have complex numbers come into our solution from both the eigenvalue and the eigenvector. Getting rid of the complex numbers here will be similar to how we did it back in the second order differential equation case, but will involve a little more work this time around. It's easiest to see how to do this in an example.

Example 1 Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Solution
We first need the eigenvalues and eigenvectors for the matrix.

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -9 \\ 4 & -3 - \lambda \end{vmatrix}$$

$$= \lambda^2 + 27 \qquad\qquad \lambda_{1,2} = \pm 3\sqrt{3}\,i$$

So, now that we have the eigenvalues recall that we only need to get the eigenvector for one of the eigenvalues since we can get the second eigenvector for free from the first eigenvector.

$\lambda_1 = 3\sqrt{3}\,i$:

We need to solve the following system.

$$\begin{pmatrix} 3-3\sqrt{3}\,i & -9 \\ 4 & -3-3\sqrt{3}\,i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the first equation we get,

$$\left(3-3\sqrt{3}\,i\right)\eta_1 - 9\eta_2 = 0$$

$$\eta_2 = \frac{1}{3}\left(1-\sqrt{3}\,i\right)\eta_1$$

So, the first eigenvector is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \dfrac{1}{3}\left(1-\sqrt{3}\,i\right)\eta_1 \end{pmatrix}$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 3 \\ 1-\sqrt{3}\,i \end{pmatrix} \qquad \eta_1 = 3$$

When finding the eigenvectors in these cases make sure that the complex number appears in the numerator of any fractions since we'll need it in the numerator later on. Also try to clear out any fractions by appropriately picking the constant. This will make our life easier down the road.

Now, the second eigenvector is,

$$\vec{\eta}^{(2)} = \begin{pmatrix} 3 \\ 1+\sqrt{3}\,i \end{pmatrix}$$

However, as we will see we won't need this eigenvector.

The solution that we get from the first eigenvalue and eigenvector is,

$$\vec{x}_1(t) = \mathbf{e}^{3\sqrt{3}\,it}\begin{pmatrix} 3 \\ 1-\sqrt{3}\,i \end{pmatrix}$$

So, as we can see there are complex numbers in both the exponential and vector that we will need to get rid of in order to use this as a solution. Recall from the complex roots section of the second order differential equation chapter that we can use [Euler's formula](#) to get the complex number out of the exponential. Doing this gives us,

$$\vec{x}_1(t) = \left(\cos\left(3\sqrt{3}t\right) + i\sin\left(3\sqrt{3}t\right)\right)\begin{pmatrix} 3 \\ 1-\sqrt{3}\,i \end{pmatrix}$$

The next step is to multiply the cosines and signs into the vector.

$$\vec{x}_1(t) = \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) + 3i\sin\left(3\sqrt{3}t\right) \\ \cos\left(3\sqrt{3}t\right) + i\sin\left(3\sqrt{3}t\right) - \sqrt{3}\,i\cos\left(3\sqrt{3}t\right) + \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix}$$

Now combine the terms with an "*i*" in them and split these terms off from those terms that don't contain an "*i*". Also factor the "*i*" out of this vector.

$$\vec{x}_1(t) = \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ \cos\left(3\sqrt{3}t\right) + \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} + i\begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ \sin\left(3\sqrt{3}t\right) - \sqrt{3}\cos\left(3\sqrt{3}t\right) \end{pmatrix}$$

$$= \vec{u}(t) + i\vec{v}(t)$$

Now, it can be shown (we'll leave the details to you) that $\vec{u}(t)$ and $\vec{v}(t)$ are two linearly independent solutions to the system of differential equations. This means that we can use them to form a general solution and they are both real solutions.

So, the general solution to a system with complex roots is

$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$$

where $\vec{u}(t)$ and $\vec{v}(t)$ are found by writing the first solution as

$$\vec{x}(t) = \vec{u}(t) + i\vec{v}(t)$$

For our system then, the general solution is,

$$\vec{x}(t) = c_1\begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ \cos\left(3\sqrt{3}t\right) + \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} + c_2\begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ \sin\left(3\sqrt{3}t\right) - \sqrt{3}\cos\left(3\sqrt{3}t\right) \end{pmatrix}$$

We now need to apply the initial condition to this to find the constants.

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}$$

This leads to the following system of equations to be solved,

$$\left.\begin{array}{c} 3c_1 = 2 \\ c_1 - \sqrt{3}c_2 = -4 \end{array}\right\} \qquad \Rightarrow \qquad c_1 = \frac{2}{3},\ c_2 = \frac{14}{3\sqrt{3}}$$

The actual solution is then,

$$\vec{x}(t) = \frac{2}{3}\begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ \cos\left(3\sqrt{3}t\right) + \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} + \frac{14}{3\sqrt{3}}\begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ \sin\left(3\sqrt{3}t\right) - \sqrt{3}\cos\left(3\sqrt{3}t\right) \end{pmatrix}$$

As we did in the last section we'll do the phase portraits separately from the solution of the system in case phase portraits haven't been taught in your class.

Example 2 Sketch the phase portrait for the system.

$$\vec{x}' = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix}\vec{x}$$
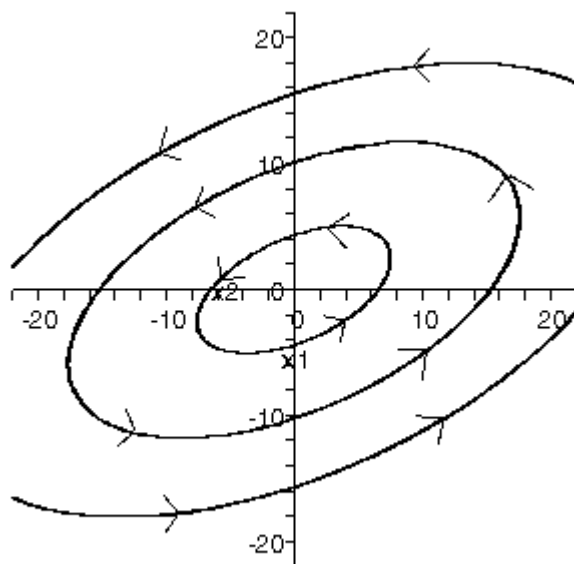
Solution

When the eigenvalues of a matrix *A* are purely complex, as they are in this case, the trajectories of the solutions will be circles or ellipses that are centered at the origin. The only thing that we really need to concern ourselves with here are whether they are rotating in a clockwise or counterclockwise direction.

This is easy enough to do. Recall when we first looked at these phase portraits a couple of sections <u>ago</u> that if we pick a value of $\vec{x}(t)$ and plug it into out system we will get a vector that will be tangent to the trajectory at that point and pointing in the direction that the trajectory is traveling.. So, let's pick the following point and see what we get.

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Therefore at the point (1,0) in the phase plane the trajectory will be point in a upwards direction. The only way that this can be is if the trajectories are traveling in a counterclockwise direction.

Here is the sketch of some of the trajectories for this problem.

The equilibrium solution in the case is called a **center** and is stable.

Note in this last example that the equilibrium solution is stable and not asymptotically stable. Asymptotically stable refers to the fact that the trajectories are moving in toward the equilibrium solution as *t* increases. In this example the trajectories are simply revolving around the equilibrium solution and not moving in towards it. The trajectories are also not moving away from the equilibrium solution and so they aren't unstable. Therefore we call the equilibrium solution stable.

Not all complex eigenvalues will result in centers so let's take a look at an example where we get something different.

Example 3 Solve the following IVP.
$$\vec{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \vec{x} \qquad\qquad \vec{x}(0) = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$$

Solution
Let's get the eigenvalues and eigenvectors for the matrix.
$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -13 \\ 5 & 1 - \lambda \end{vmatrix}$$
$$= \lambda^2 - 4\lambda + 68 \qquad\qquad \lambda_{1,2} = 2 \pm 8i$$

Now get the eigenvector for the first eigenvalue.

$\lambda_1 = 2 + 8i$:
We need to solve the following system.
$$\begin{pmatrix} 1 - 8i & -13 \\ 5 & -1 - 8i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the second equation we get,
$$5\eta_1 + (-1 - 8i)\eta_2 = 0$$
$$\eta_1 = \frac{1}{5}(1 + 8i)\eta_2$$

So, the first eigenvector is,
$$\vec{\eta} = \begin{pmatrix} \dfrac{1}{5}(1 + 8i)\eta_2 \\ \eta_2 \end{pmatrix}$$
$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 + 8i \\ 5 \end{pmatrix} \qquad \eta_2 = 5$$

The solution corresponding the this eigenvalue and eigenvector is
$$\vec{x}_1(t) = \mathbf{e}^{(2+8i)t} \begin{pmatrix} 1 + 8i \\ 5 \end{pmatrix}$$
$$= \mathbf{e}^{2t} \mathbf{e}^{8it} \begin{pmatrix} 1 + 8i \\ 5 \end{pmatrix}$$
$$= \mathbf{e}^{2t} \left(\cos(8t) + i\sin(8t) \right) \begin{pmatrix} 1 + 8i \\ 5 \end{pmatrix}$$

As with the first example multiply cosines and signs into the vector and split it up. Don't forget about the exponential that is in the solution this time.

$$\vec{x}_1(t) = \mathbf{e}^{2t}\begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + i\mathbf{e}^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix}$$

$$= \vec{u}(t) + i\,\vec{v}(t)$$

The general solution to this system then,

$$\vec{x}(t) = c_1\mathbf{e}^{2t}\begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + c_2\mathbf{e}^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix}$$

Now apply the initial condition and find the constants.

$$\begin{pmatrix} 3 \\ -10 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2\mathbf{e}^{2t}\begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{r} c_1 + 8c_2 = 3 \\ 5c_1 = -10 \end{array}\right\} \qquad \Rightarrow \qquad c_1 = -2,\ c_2 = \frac{5}{8}$$

The actual solution is then,

$$\vec{x}(t) = -2\mathbf{e}^{2t}\begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + \frac{5}{8}\mathbf{e}^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix}$$

Let's take a look at the phase portrait for this problem.

Example 4 Sketch the phase portrait for the system.

$$\vec{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}\vec{x}$$

Solution
When the eigenvalues of a system are complex with a real part the trajectories will spiral into or out of the origin. We can determine which one it will be by looking at the real portion. Since the real portion will end up being the exponent of an exponential function (as we saw in the solution to this system) if the real part is positive the solution will grow very large as *t* increases. Likewise, if the real part is negative the solution will die out as *t* increases.
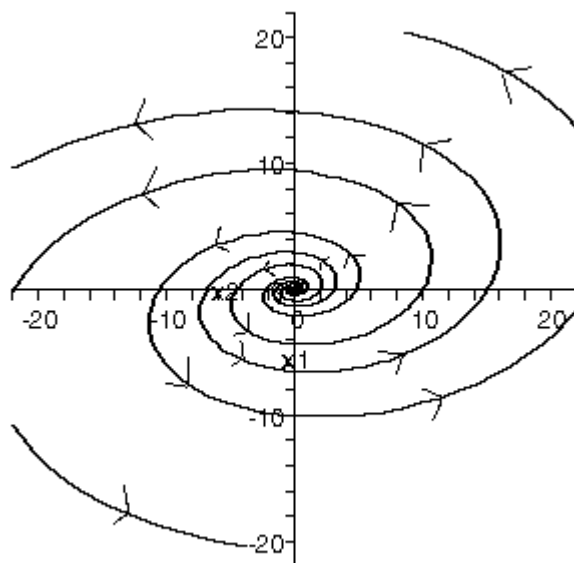
So, if the real part is positive the trajectories will spiral out from the origin and if the real part is negative they will spiral into the origin. We determine the direction of rotation (clockwise vs. counterclockwise) in the same way that we did for the center.

In our case the trajectories will spiral out from the origin since the real part is positive and

$$\vec{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

will rotate in the counterclockwise direction as the last example did.

Here is a sketch of some of the trajectories for this system.

Here we call the equilibrium solution a **spiral** (oddly enough…) and in this case it's unstable since the trajectories move away from the origin.

If the real part of the eigenvalue is negative the trajectories will spiral into the origin and in this case the equilibrium solution will be asymptotically stable.

Repeated Eigenvalues

This is the final case that we need to take a look at. In this section we are going to look at solutions to the system,

$$\vec{x}' = A\vec{x}$$

where the eigenvalues are repeated eigenvalues. Since we are going to be working with systems in which *A* is a 2 x 2 matrix we will make that assumption from the start. So the system will have a double eigenvalue, λ.

This presents us with a problem. We want two linearly independent solutions so that we can form a general solution. However, with a double eigenvalue we will have only one,

$$\vec{x}_1 = \vec{\eta}\mathbf{e}^{\lambda t}$$

So, we need to come up with a second solution. Recall that when we looked at the double root case with the second order differential equations we ran into a similar problem. In that section we simply added a *t* to the solution and were able to get a second solution. Let's see if the same thing will work in this case as well. We'll see if

$$\vec{x} = t\,\mathbf{e}^{\lambda t}\vec{\eta}$$

will also be a solution.

To check all we need to do is plug into the system. Don't forget to product rule the proposed solution when you differentiate!

$$\vec{\eta}\mathbf{e}^{\lambda t} + \lambda\vec{\eta}t\mathbf{e}^{\lambda t} = A\vec{\eta}t\mathbf{e}^{\lambda t}$$

Now, we got two functions here on the left side, an exponential by itself and an exponential times a t. So, in order for our guess to be a solution we will need to require,

$$A\vec{\eta} = \lambda\vec{\eta} \qquad \Rightarrow \qquad (A - \lambda I)\vec{\eta} = \vec{0}$$
$$\vec{\eta} = \vec{0}$$

The first requirement isn't a problem since we this just says that λ is an eigenvalue and it's eigenvector is $\vec{\eta}$. We already knew this however so there's nothing new there. The second however is a problem. Since $\vec{\eta}$ is an eigenvalue we know that it can't be zero, yet in order to satisfy the second condition it would have to be.

So, our guess was incorrect. The problem seems to be that there is a lone term with just an exponential in it so let's see if we can't fix up our guess to correct that. Let's try the following guess.

$$\vec{x} = t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}$$

where $\vec{\rho}$ is an unknown vector that we'll need to determine.

As with the first guess let's plug this into the system and see what we get.

$$\vec{\eta}\mathbf{e}^{\lambda t} + \lambda\vec{\eta}t\mathbf{e}^{\lambda t} + \lambda\vec{\rho}\mathbf{e}^{\lambda t} = A\left(\vec{\eta}t\mathbf{e}^{\lambda t} + \vec{\rho}\mathbf{e}^{\lambda t}\right)$$
$$\left(\vec{\eta} + \lambda\vec{\rho}\right)\mathbf{e}^{\lambda t} + \lambda\vec{\eta}t\mathbf{e}^{\lambda t} = A\vec{\eta}t\mathbf{e}^{\lambda t} + A\vec{\rho}\mathbf{e}^{\lambda t}$$

Now set coefficients equal again,

$$\lambda\vec{\eta} = A\vec{\eta} \qquad \Rightarrow \qquad (A - \lambda I)\vec{\eta} = \vec{0}$$
$$\vec{\eta} + \lambda\vec{\rho} = A\vec{\rho} \qquad \Rightarrow \qquad (A - \lambda I)\vec{\rho} = \vec{\eta}$$

As with our first guess the first equation tells us nothing that we didn't already know. This time the second equation is not a problem. All the second equation tells us is that $\vec{\rho}$ must be a solution to this equation.

It looks like our second guess worked. Therefore,

$$\vec{x}_2 = t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}$$

will be a solution to the system provided $\vec{\rho}$ is a solution to

$$(A - \lambda I)\vec{\rho} = \vec{\eta}$$

Also this solution and the first solution are linearly independent and so they form a fundamental set of solutions and so the general solution in the double eigenvalue case is,

$$\vec{x} = c_1\mathbf{e}^{\lambda t}\vec{\eta} + c_2\left(t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}\right)$$

Let's work an example.

Example 1 Solve the following IVP.
$$\vec{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \vec{x} \qquad\qquad \vec{x}(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

Solution
First find the eigenvalues for the system.
$$\det(A - \lambda I) = \begin{vmatrix} 7 - \lambda & 1 \\ -4 & 3 - \lambda \end{vmatrix}$$
$$= \lambda^2 - 10\lambda + 25$$
$$= (\lambda - 5)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = 5$$

So, we got a double eigenvalue. Of course that shouldn't be too surprising given the section that we're in. Let's find the eigenvector for this eigenvalue.
$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad 2\eta_1 + \eta_2 = 0 \qquad \eta_2 = -2\eta_1$$

The eigenvector is then,
$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -2\eta_1 \end{pmatrix} \qquad\qquad \eta_1 \neq 0$$
$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad\qquad \eta_1 = 1$$

The next step is find $\vec{\rho}$. To do this we'll need to solve,
$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \Rightarrow \qquad 2\rho_1 + \rho_2 = 1 \qquad \rho_2 = 1 - 2\rho_1$$

Note that this is almost identical to the system that we solve to find the eigenvalue. The only difference is the right hand side. The most general possible $\vec{\rho}$ is
$$\vec{\rho} = \begin{pmatrix} \rho_1 \\ 1 - 2\rho_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\rho} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{if } \rho_1 = 0$$

In this case, unlike the eigenvector system we can choose the constant to be anything we want, so we might as well pick it to make our life easier. This usually means picking it to be zero.

We can now write down the general solution to the system.
$$\vec{x}(t) = c_1 \mathbf{e}^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \left(\mathbf{e}^{5t}t\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \mathbf{e}^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Applying the initial condition to find the constants gives us,
$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$c_1 = 2 \atop -2c_1 + c_2 = -5 \Bigg\} \qquad \Rightarrow \qquad c_1 = 2, \ c_2 = -1$$

The actual solution is then,

$$\vec{x}(t) = 2\mathbf{e}^{5t}\begin{pmatrix}1\\-2\end{pmatrix} - \left(t\mathbf{e}^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + \mathbf{e}^{5t}\begin{pmatrix}0\\1\end{pmatrix} \right)$$

$$= \mathbf{e}^{5t}\begin{pmatrix}2\\-4\end{pmatrix} - \mathbf{e}^{5t}t\begin{pmatrix}1\\-2\end{pmatrix} - \mathbf{e}^{5t}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \mathbf{e}^{5t}\begin{pmatrix}2\\-5\end{pmatrix} - \mathbf{e}^{5t}t\begin{pmatrix}1\\-2\end{pmatrix}$$

Note that we did a little combining here to simplify the solution up a little.

So, the next example will be to sketch the phase portrait for this system.

*Example 2* Sketch the phase portrait for the system.
$$\vec{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\vec{x}$$

Solution
These will start in the same way that real, distinct eigenvalue phase portraits start. We'll first sketch in a trajectory that is parallel to the eigenvector and note that since the eigenvalue is positive the trajectory will be moving away from the origin.
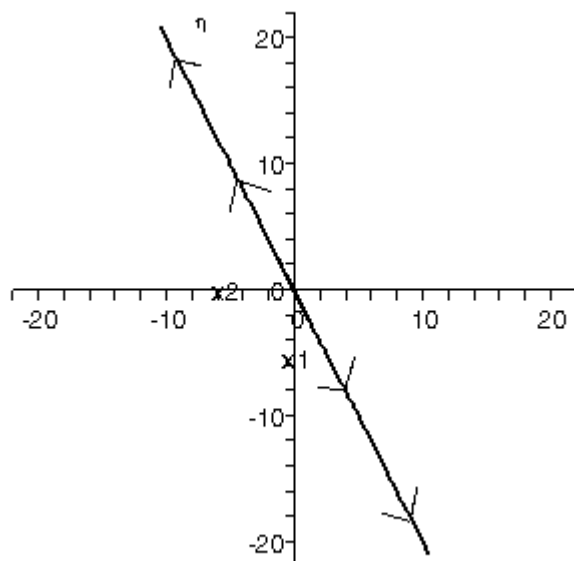
Now, it will be easier to explain the remainder of the phase portrait if we actually have one in front of us. So here is the full phase portrait with some more trajectories sketched in.
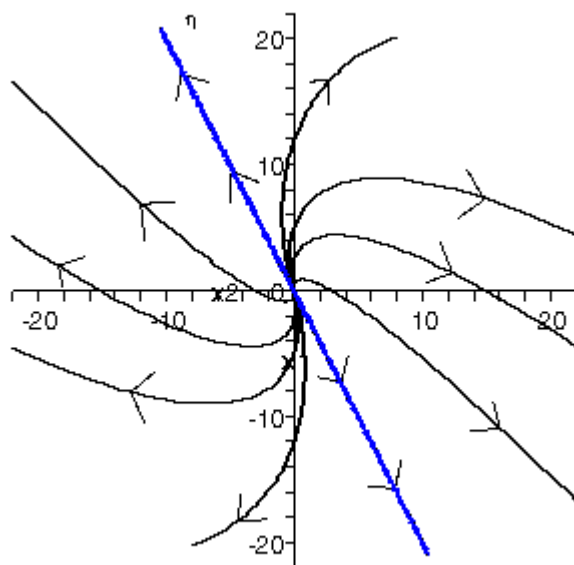
Trajectories in these cases always emerge from (or move into) the origin in a direction that is parallel to the eigenvector. Likewise they will start in one direction before turning around and moving off into the other direction. The directions in which they move are opposite depending on which side of the trajectory corresponding to the eigenvector we are on. Also, as the trajectories moves away from the origin it should start becoming parallel to the trajectory corresponding to the eigenvector.

So, how do we determine the direction? We can do the same thing that we did in the complex case. We'll plug in (1,0) into the system and see which direction the trajectories are moving at that point. Since this point is directly to the right of the origin the trajectory at that point must have already turned around and so this will give the direction that it will traveling after turning around.

Doing that for this problem to check our phase portrait gives,

$$\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

This vector will point down into the fourth quadrant and so the trajectory must be moving into the fourth quadrant as well. This does match up with our phase portrait.

In these cases the equilibrium is called a **node** and is unstable in this case. Note that sometimes you will hear nodes for the repeated eigenvalue case called **degenerate nodes** or **improper nodes**.

Let's work one more example.

Example 3 Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix} \vec{x} \qquad\qquad \vec{x}(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution

First the eigenvalue for the system.

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & \frac{3}{2} \\ -\frac{1}{6} & -2-\lambda \end{vmatrix}$$

$$= \lambda^2 + 3\lambda + \frac{9}{4}$$

$$= \left(\lambda + \frac{3}{2}\right)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = -\frac{3}{2}$$

Now let's get the eigenvector.

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \frac{1}{2}\eta_1 + \frac{3}{2}\eta_2 = 0 \qquad \eta_1 = -3\eta_2$$

$$\vec{\eta} = \begin{pmatrix} -3\eta_2 \\ \eta_2 \end{pmatrix} \qquad\qquad \eta_2 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad\qquad \eta_2 = 1$$

Now find $\vec{\rho}$,

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \frac{1}{2}\rho_1 + \frac{3}{2}\rho_2 = -3 \qquad \rho_1 = -6 - 3\rho_2$$

$$\vec{\rho} = \begin{pmatrix} -6 - 3\rho_2 \\ \rho_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\rho} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \qquad \text{if } \rho_2 = 0$$

The general solution for the system is then,

$$\vec{x}(t) = c_1 e^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \left(t e^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + e^{-\frac{3t}{2}} \begin{pmatrix} -6 \\ 0 \end{pmatrix} \right)$$

Applying the initial condition gives,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(2) = c_1 e^{-3} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \left(2 e^{-3} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + e^{-3} \begin{pmatrix} -6 \\ 0 \end{pmatrix} \right)$$

Note that we didn't use *t=0* this time! We now need to solve the following system,

$$\left. \begin{array}{r} -3 e^{-3} c_1 - 12 e^{-3} c_2 = 1 \\ e^{-3} c_1 + 2 e^{-3} c_2 = 0 \end{array} \right\} \qquad \Rightarrow \qquad c_1 = \frac{e^3}{3}, \ c_2 = -\frac{e^3}{6}$$

The actual solution is then,

$$\vec{x}(t) = \frac{\mathbf{e}^3}{3}\mathbf{e}^{-\frac{3t}{2}}\begin{pmatrix} -3 \\ 1 \end{pmatrix} - \frac{\mathbf{e}^3}{6}\left(t\mathbf{e}^{-\frac{3t}{2}}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{e}^{-\frac{3t}{2}}\begin{pmatrix} -6 \\ 0 \end{pmatrix} \right)$$

$$= \mathbf{e}^{-\frac{3t}{2}+3}\begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} + t\mathbf{e}^{-\frac{3t}{2}+3}\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{6} \end{pmatrix}$$

And just to be consistent with all the other problems that we've done let's sketch the phase portrait.

***Example 4*** Sketch the phase portrait for the system.

$$\vec{x}' = \begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix}\vec{x}$$

Solution
Let's first notice that since the eigenvalue is negative in this case the trajectories should all move in towards the origin. Let's check the direction of the trajectories at (1,0)

$$\begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{1}{6} \end{pmatrix}$$

So it looks like the trajectories should be pointing into the third quadrant at (1,0). This gives the following phase portrait.
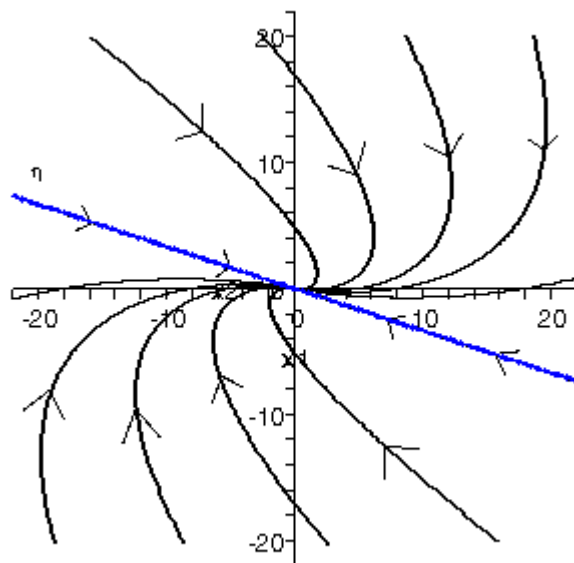
Nonhomogeneous Systems

We now need to address nonhomogeneous systems briefly. Both of the methods that we looked at back in the second order differential equations chapter can also be used here. As we will see Undetermined Coefficients is almost identical when used on systems while Variation of Parameters will need to have a new formula derived, but will actually be slightly easier when applied to systems.

Undetermined Coefficients
Undetermined coefficients for systems is pretty much identical to the second order differential equation case. The only difference is that the coefficients will need to be vectors now.

Let's take a quick look at an example.

Example 1 Find the general solution to the following system.
$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x} + t \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Solution
We already have the complimentary solution as we solved that part back in the real eigenvalue section. It is,
$$\vec{x}_c(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Guessing the form of the particular solution will work in exactly the same way it did back when we first looked at this method. We have a linear polynomial and so our guess will need to be a linear polynomial. The only difference is that the "coefficients" will need to be vectors instead of constants. The particular solution will have the form,
$$\vec{x}_P = t\vec{a} + \vec{b} = t \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

So, we need to differentiate the guess
$$\vec{x}_P' = \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Before plugging into the system let's simplify the notation a little to help with our work. We'll write the system as,
$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x} + t \begin{pmatrix} 2 \\ -4 \end{pmatrix} = A\vec{x} + t\vec{g}$$

This will make the following work a little easier. Now, let's plug things into the system.

$$\vec{a} = A\left(t\vec{a} + \vec{b}\right) + t\vec{g}$$

$$\vec{a} = tA\vec{a} + A\vec{b} + t\vec{g}$$

$$\vec{0} = t\left(A\vec{a} + \vec{g}\right) + \left(A\vec{b} - \vec{a}\right)$$

Now we need to set the coefficients equal. Doing this gives,

$$t^1: \qquad A\vec{a} + \vec{g} = \vec{0} \qquad\qquad A\vec{a} = -\vec{g}$$

$$t^0: \qquad A\vec{b} - \vec{a} = \vec{0} \qquad\qquad A\vec{b} = \vec{a}$$

Now only $\vec{a}$ is unknown in the first equation so we can use Gaussian elimination to solve the system. We'll leave this work to you to check.

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\begin{pmatrix} 2 \\ -4 \end{pmatrix} \qquad \Rightarrow \qquad \vec{a} = \begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix}$$

Now that we know $\vec{a}$ we can solve the second equation for $\vec{b}$.

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} \qquad \Rightarrow \qquad \vec{b} = \begin{pmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{pmatrix}$$

So, since we were able to solve both equations, the particular solution is then,

$$\vec{x}_P = t\begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} + \begin{pmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{pmatrix}$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} + \begin{pmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{pmatrix}$$

So, as you can see undetermined coefficients is nearly the same as the first time we saw it. The work in solving for the "constants" is a little messier however.

Variation of Parameters
In this case we will need to derive a new formula for variation of parameters for systems. The derivation this time will be much simpler than the when we first saw variation of parameters.

First let $X(t)$ be a matrix whose i^{th} column is the ith linearly independent solution to the system,

$$\vec{x}' = A\vec{x}$$

Now it can be shown that $X(t)$ will be a solution to the following differential equation.

$$X' = AX \qquad\qquad\qquad (1)$$

This is nothing more than the original system with the matrix in place of the original vector.

We are going to try and find a particular solution to
$$\vec{x}' = A\vec{x} + \vec{g}(t)$$
We will assume that we can find a solution of the form,
$$\vec{x}_P = X(t)\vec{v}(t)$$
where we will need to determine the vector $\vec{v}(t)$. To do this we will need to plug this into the nonhomogeneous system. Don't forget to product rule the particular solution when plugging the guess into the system.
$$X'\vec{v} + X\vec{v}' = A X \vec{v} + \vec{g}$$

Note that we dropped the "*(t)*" part of things to simplify the notation a little. Now using (1) we can rewrite this a little.
$$X'\vec{v} + X\vec{v}' = X'\vec{v} + \vec{g}$$
$$X\vec{v}' = \vec{g}$$

Because we formed *X* using linearly independent solutions we know that det(*X*) must be nonzero and this in turn means that we can find the inverse of *X*. So, multiply both sides by the inverse of *X*.
$$\vec{v}' = X^{-1}\vec{g}$$

Now all that we need to do is integrate both sides to get $\vec{v}(t)$.
$$\vec{v}(t) = \int X^{-1}\vec{g}\,dt$$

As with the second order differential equation case we can ignore any constants of integration. The particular solution is then,
$$\vec{x}_P = X\int X^{-1}\vec{g}\,dt \qquad\qquad \textbf{(2)}$$

Let's work a quick example using this.

Example 2 Find the general solution to the following system.
$$\vec{x}' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}\vec{x} + \mathbf{e}^{2t}\begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

Solution
We found the complimentary solution to this system in the real eigenvalue section. It is,
$$\vec{x}_c(t) = c_1\mathbf{e}^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2\mathbf{e}^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now the matrix *X* is,

$$X = \begin{pmatrix} \mathbf{e}^{-t} & -\mathbf{e}^{-6t} \\ 4\mathbf{e}^{-t} & \mathbf{e}^{-6t} \end{pmatrix}$$

Now, we need to find the inverse of this matrix. We saw how to find inverses of matrices back in the second linear algebra review [section](#) and the process is the same here even though we don't have constant entries. We'll leave the detail to you to check.

$$X^{-1} = \begin{pmatrix} \tfrac{1}{5}\mathbf{e}^{t} & \tfrac{1}{5}\mathbf{e}^{t} \\ -\tfrac{4}{5}\mathbf{e}^{6t} & \tfrac{1}{5}\mathbf{e}^{6t} \end{pmatrix}$$

Now do the multiplication in the integral.

$$X^{-1}\vec{g} = \begin{pmatrix} \tfrac{1}{5}\mathbf{e}^{t} & \tfrac{1}{5}\mathbf{e}^{t} \\ -\tfrac{4}{5}\mathbf{e}^{6t} & \tfrac{1}{5}\mathbf{e}^{6t} \end{pmatrix}\begin{pmatrix} 6\mathbf{e}^{2t} \\ -\mathbf{e}^{2t} \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{3t} \\ -5\mathbf{e}^{8t} \end{pmatrix}$$

Now do the integral.

$$\int X^{-1}\vec{g}\,dt = \int \begin{pmatrix} \mathbf{e}^{3t} \\ -5\mathbf{e}^{8t} \end{pmatrix} dt = \begin{pmatrix} \int \mathbf{e}^{3t}\,dt \\ \int -5\mathbf{e}^{8t}\,dt \end{pmatrix} = \begin{pmatrix} \tfrac{1}{3}\mathbf{e}^{3t} \\ -\tfrac{5}{8}\mathbf{e}^{8t} \end{pmatrix}$$

Remember that to integrate a matrix or vector you just integrate the individual entries.

We can now get the particular solution.

$$\begin{aligned}
\vec{x}_P &= X \int X^{-1}\vec{g}\,dt \\
&= \begin{pmatrix} \mathbf{e}^{-t} & -\mathbf{e}^{-6t} \\ 4\mathbf{e}^{-t} & \mathbf{e}^{-6t} \end{pmatrix}\begin{pmatrix} \tfrac{1}{3}\mathbf{e}^{3t} \\ -\tfrac{5}{8}\mathbf{e}^{8t} \end{pmatrix} \\
&= \begin{pmatrix} \tfrac{23}{24}\mathbf{e}^{2t} \\ \tfrac{17}{24}\mathbf{e}^{2t} \end{pmatrix} \\
&= \mathbf{e}^{2t}\begin{pmatrix} \tfrac{23}{24} \\ \tfrac{17}{24} \end{pmatrix}
\end{aligned}$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 \mathbf{e}^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathbf{e}^{2t}\begin{pmatrix} \tfrac{23}{24} \\ \tfrac{17}{24} \end{pmatrix}$$

So, some of the work can be a little messy, but overall not to bad.

We looked at two methods of solving nonhomogeneous differential equations here and while the work can be a little messy they aren't too bad. Of course we also kept the nonhomogeneous part fairly simple here. More complicated problems will have significant amounts of work involved.

Laplace Transforms

There's not too much to this section. We're just going to work an example to illustrate how Laplace transforms can be used to solve systems of differential equations.

Example 1 Solve the following system.

$$x_1' = 3x_1 - 3x_2 + 2 \qquad\qquad x_1(0) = 1$$
$$x_2' = -6x_1 - t \qquad\qquad x_2(0) = -1$$

Solution

First notice that the system is not given in matrix form. This is because the system won't be solved in matrix form. Also note that the system is nonhomogeneous.

We start just as we did when we used Laplace transforms to solve single differential equations. We take the transform of both differential equations.

$$sX_1(s) - x_1(0) = 3X_1(s) - 3X_2(s) + \frac{2}{s}$$

$$sX_2(s) - x_2(0) = -6X_1(s) - \frac{1}{s^2}$$

Now plug in the initial condition and simplify things a little.

$$(s-3)X_1(s) + 3X_2(s) = \frac{2}{s} + 1 = \frac{2+s}{s}$$

$$6X_1(s) + sX_2(s) = -\frac{1}{s^2} - 1 = -\frac{s^2+1}{s^2}$$

Now we need to solve this for one of the transforms. We'll do this by multiplying the top equation by *s* and the bottom by -3 and then adding. This gives,

$$(s^2 - 3s - 18)X_1(s) = 2 + s + \frac{3s^2 + 3}{s^2}$$

Solving for *X₁* gives,

$$X_1(s) = \frac{s^3 + 5s^2 + 3}{s^2(s+3)(s-6)}$$

Partial fractioning gives,

$$X_1(s) = \frac{1}{108}\left(\frac{133}{s-6} - \frac{28}{s+3} + \frac{3}{s} - \frac{18}{s^2}\right)$$

Taking the inverse transform gives us the first solution,

$$x_1(t) = \frac{1}{108}\left(133\mathbf{e}^{6t} - 28\mathbf{e}^{-3t} + 3 - 18t\right)$$

Now to find the second solution we could go back up and eliminate X_1 to find the transform for X_2 and sometimes we would need to do that. However, in this case notice

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that the second differential equation is,
$$x_2' = -6x_1 - t \qquad \Rightarrow \qquad x_2 = \int -6x_1 - t \, dt$$

So, plugging the first solution in and integrating gives,
$$x_2(t) = -\frac{1}{18} \int 133\mathbf{e}^{6t} - 28\mathbf{e}^{-3t} + 3 \, dt$$
$$= -\frac{1}{108}\left(133\mathbf{e}^{6t} + 56\mathbf{e}^{-3t} + 18t\right) + c$$

Now, reapplying the second initial condition to get the constant of integration gives
$$-1 = -\frac{1}{108}(133 + 56) + c \qquad \Rightarrow \qquad c = \frac{3}{4}$$

The second solution is then,
$$x_2(t) = -\frac{1}{108}\left(133\mathbf{e}^{6t} + 56\mathbf{e}^{-3t} + 18t - 81\right)$$

So, putting all this together gives the solution to the system as,
$$x_1(t) = \frac{1}{108}\left(133\mathbf{e}^{6t} - 28\mathbf{e}^{-3t} + 3 - 18t\right)$$
$$x_2(t) = -\frac{1}{108}\left(133\mathbf{e}^{6t} + 56\mathbf{e}^{-3t} + 18t - 81\right)$$

Compared to the last section the work here wasn't too bad. That won't always be the case of course, but you can see that using Laplace transforms to solve systems isn't too bad in at least some cases.

   http://tutorial.math.lamar.edu/terms.asp